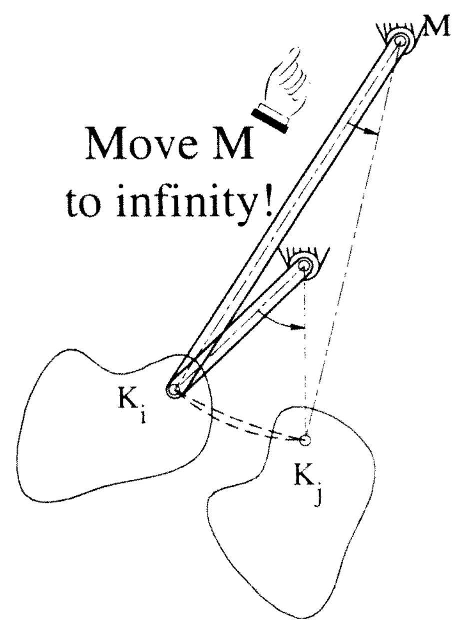
Take another thoughtful look at the compatibility equation:

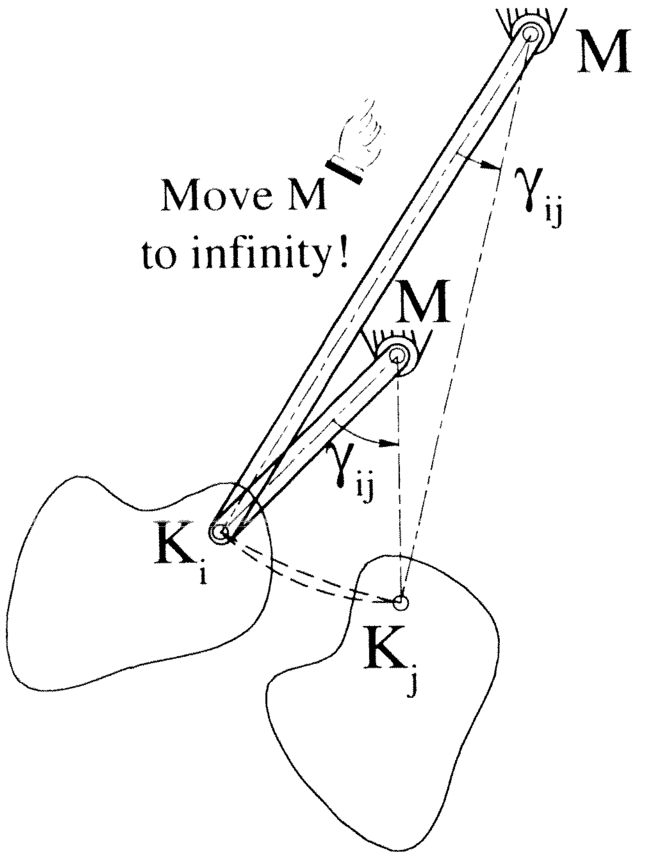
$$\begin{vmatrix} \left(e^{i\phi_2}-1\right) & \left(e^{i\gamma_2}-1\right) & \boldsymbol{\delta}_2 \\ \left(e^{i\phi_3}-1\right) & \left(e^{i\gamma_3}-1\right) & \boldsymbol{\delta}_3 \\ \left(e^{i\phi_4}-1\right) & \left(e^{i\gamma_4}-1\right) & \boldsymbol{\delta}_4 \end{vmatrix} = 0$$

Notice that this equation is trivially satisfied if all the γ_j's $\equiv 0$. In this case, the center column becomes all zeros. But how can this be a solution??? After all, the body is moving through four finitely separated positions. Having all the gammas equal to zero means that the crank doesn't rotate as the body moves from one position to another!

HOWZITPOZZIBLE??????

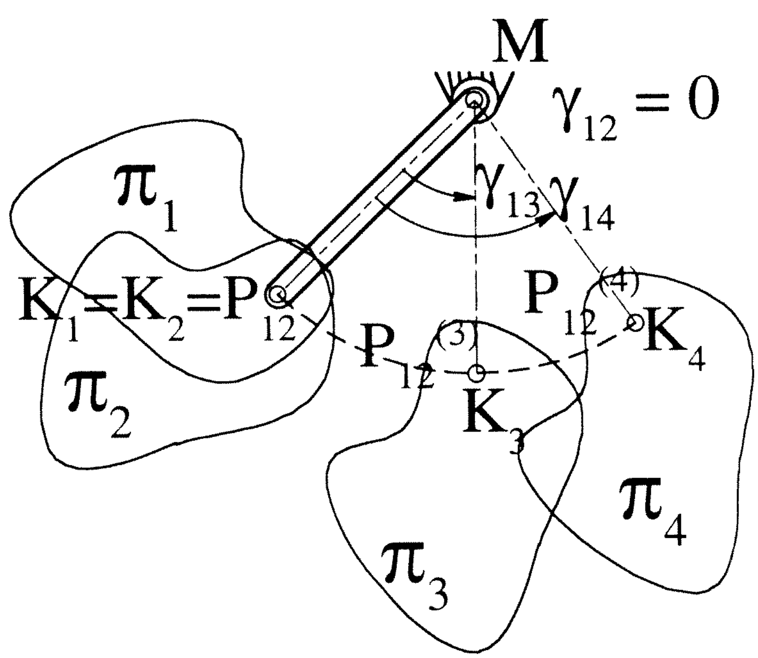
I'll tell you howzitpozzible. The centerpoint M is at infinity so the crank is infinitely long! An infinitesimal twitch of this infinitely long link moves the circlepoint K through a finite displacement along a straight line (since the radius is infinite). Point K is a sliderpoint and can be used for designing various types of slider mechanisms for the four given positions.





As point M moves further and further out, the swing angle γ of the link gets smaller and smaller even though the circlepoint K is still moving through a finite displacement. At the same time, the arc along which K moves gets flatter and flatter as its radius increases. In the limit, the arc becomes a straight line and the swing angle becomes zero. At that point, it is a lot cheaper to use a slider to move point K from its i^{th} position to its j^{th} position than it is to hire a rental car to take a union workman out to infinity to install and maintain a fixed pivot there.

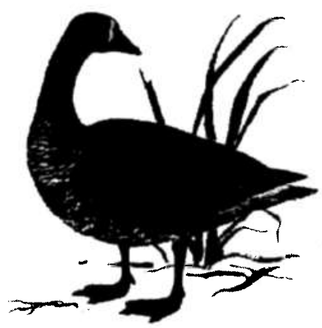
Here's another neat thing to notice: It is possible to have one of the gammas in the compatibility equation be zero and not have the others be zero at the same time. What that means is that the circlepoint K is superimposed with a pole for the two positions in question and that the four position solu-



$P_{12}^{(3)}$

tion has degenerated into a special case of a three position problem. This is what Kurt Hain calls “point position reduction”. The link will dwell somewhat as it moves between the two positions and then start to swing again as it moves on to the remaining design positions.

In the example shown here, the angle γ_{12} is equal to zero so the point K falls on top of the pole P_{12}. The link shown has no swing when the body is in either position one or two, but it does swing as the body goes on to positions three and four. Actually, the link might move while the link is passing between positions one and two, since the synthesis only controls what happens in the design positions and not what happens as the mechanism moves between positions.



Gander again at the compatibility equation:

$$\begin{vmatrix} \left(e^{i\phi_2}-1\right) & \left(e^{i\gamma_2}-1\right) & \boldsymbol{\delta}_2 \\ \left(e^{i\phi_3}-1\right) & \left(e^{i\gamma_3}-1\right) & \boldsymbol{\delta}_3 \\ \left(e^{i\phi_4}-1\right) & \left(e^{i\gamma_4}-1\right) & \boldsymbol{\delta}_4 \end{vmatrix} = 0$$

If $\gamma_2 = \phi_2$, $\gamma_3 = \phi_3$, and $\gamma_4 = \phi_4$ azzaforinstancefordasakeadiscussion, then the equation will also be trivially satisfied, because the second column will be identical to the first.

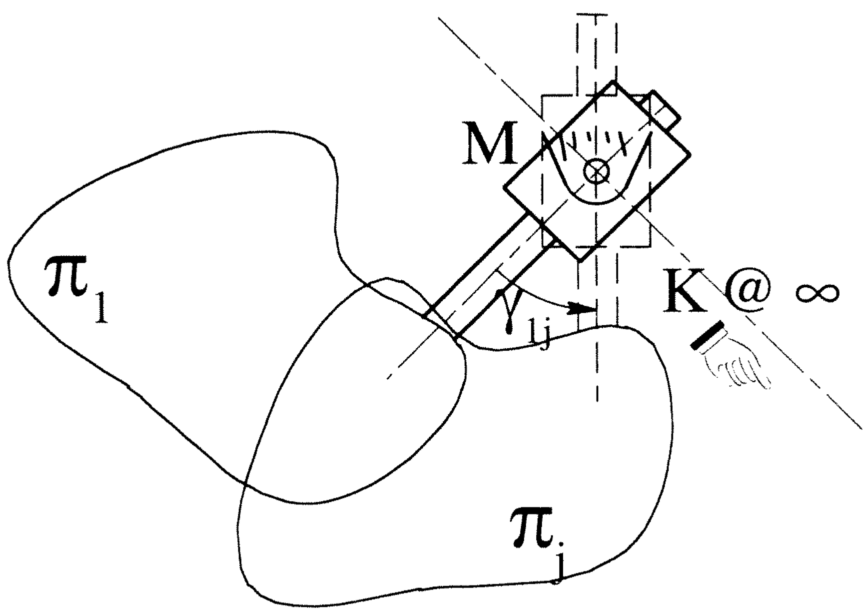
HowDyaAccountForThat??????

From a physical point of view, this means that the rotations of the crank are now the same as the rotations of the moving body. In the case we just finished studying, the rotations of the crank were the same as the rotations of the fixed body, namely zero.

☞ Notice the perfect symmetry involved. ☜

When $\gamma_j = 0$ (≡ frame rotations) we had a sliderpoint K which moved along a straight path perpendicular to the infinitely long link K—M. The centerpoint M was at infinity.

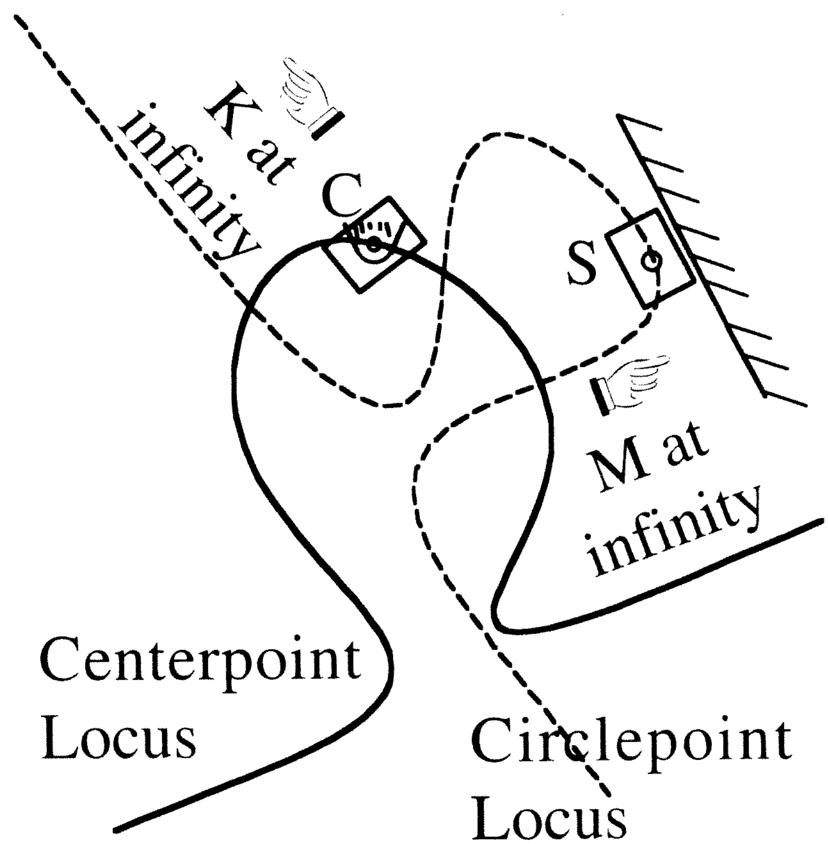
In the case we are now considering, the $\gamma_j = \phi_j$ (≡ moving body rotations). We have the identical situation but for the inverted motion! Point M is now a sliderpoint of the *inverted* motion. In other words, if an observer were riding on the moving body rather than standing on the fixed body he or she would see the circlepoint K at infinity.



As seen from the point of view of observers sitting on the fixed frame, there would be a straight line of sliding attached to the moving body. As the body moves through its various positions, this straight line would always pass through the point M on the fixed body, so we can call point M a "Concurrency Point". It can be used as a location for a slider pivoted to the frame at M and with a straight slide track in the moving body. That slide track is perpendicular to K—M in each design position. This time, however, the point K is at infinity and point M is right there in front of your nose!

If you have a case where one of the gamma's is equal to one of the phi's but the other gamma's and phi's aren't equal, then that means that the fixed pivot M is located at the pole for the two corresponding positions. This is in exact analogy with the inverse case in which the moving pivot was located at a pole.

Thus, the slider point for four positions is a circlepoint whose corresponding centerpoint lies at infinity. Conversely, the concurrency point for four positions is a centerpoint whose corresponding circlepoint lies at infinity. The slider point is pinned to the moving body and the concurrency point is pinned to the fixed body.



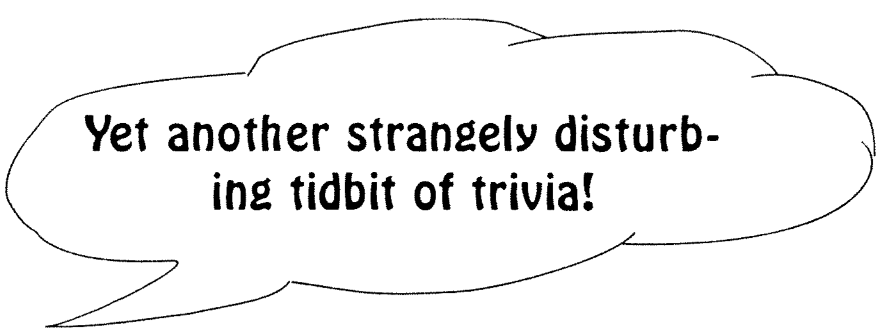
If $\gamma_2 = 0$ but γ_3 and γ_4 are not zero, then this means that the *circlepoint* is at the pole $\mathbf{P}_{12}$ and four positions have reduced down to three displacements of the circlepoint for synthesis purposes. (Point position reduction!) Similarly, if $\gamma_2 = \phi_2$ but $\gamma_3 \neq \phi_3$ and $\gamma_4 \neq \phi_4$ then the *centerpoint* is at the pole $\mathbf{P}_{12}$ and we have a different form of point position reduction.

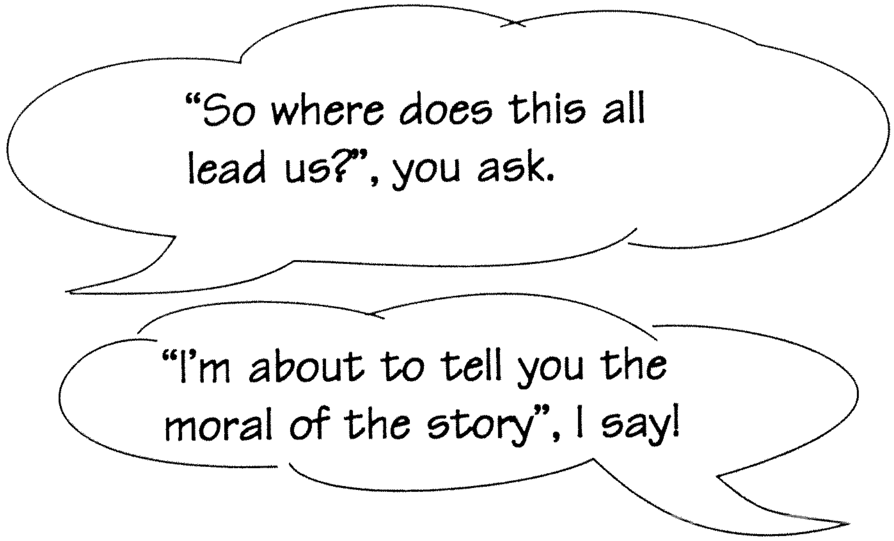
If $\gamma_j - \gamma_k = 0$ for j, k = 2, 3, or 4 with $j \neq k$ then the circlepoint will correspond to the image pole $\mathbf{P}_{jk}^{(1)}$ giving yet another wrinkle on how you can use point position reduction.

Naturally, we can arbitrarily select the numbering of our positions for convenience, so we can put any pole we want at the fixed or moving pivot.

Since the centerpoint curve passes through the six poles $\mathbf{P}_{12}$, $\mathbf{P}_{13}$, $\mathbf{P}_{14}$, $\mathbf{P}_{23}$, $\mathbf{P}_{24}$, $\mathbf{P}_{\mathbf{34}}$ it is also known as the "*pole curve*".

The circlepoint curve passes through the six poles in their position one images: $\mathbf{P}_{12}$, $\mathbf{P}_{13}$, $\mathbf{P}_{14}$, $\mathbf{P}_{23}^{(1)}$, $\mathbf{P}_{24}^{(1)}$, $\mathbf{P}_{\mathbf{34}}^{\mathbf{(1)}}$.





The moral of the story is that you can synthesize *all kinds* of different mechanisms by using various combinations of these special slider points and concurrency points along with ordinary circlepoints and centerpoints. For instance, you can design damn near any kind of a mechanism that can be built with pin or slider joints. You can make all sorts of slider-crank mechanisms of any conceivable inversion, double slider mechanisms, turning-block mechanisms, tumbling block mechanisms, four-bars, dwell mechanisms, bits and pieces of six, eight, or ten link mechanisms, frammis glitzifiers, widgets, and more once you get the hang of how to do it. To get my free booklet on how to get the hang of it send 10,000 cereal box tops along with the original sales receipts and UPS labels to:

Dr. K's Fantastic Booklet Offer
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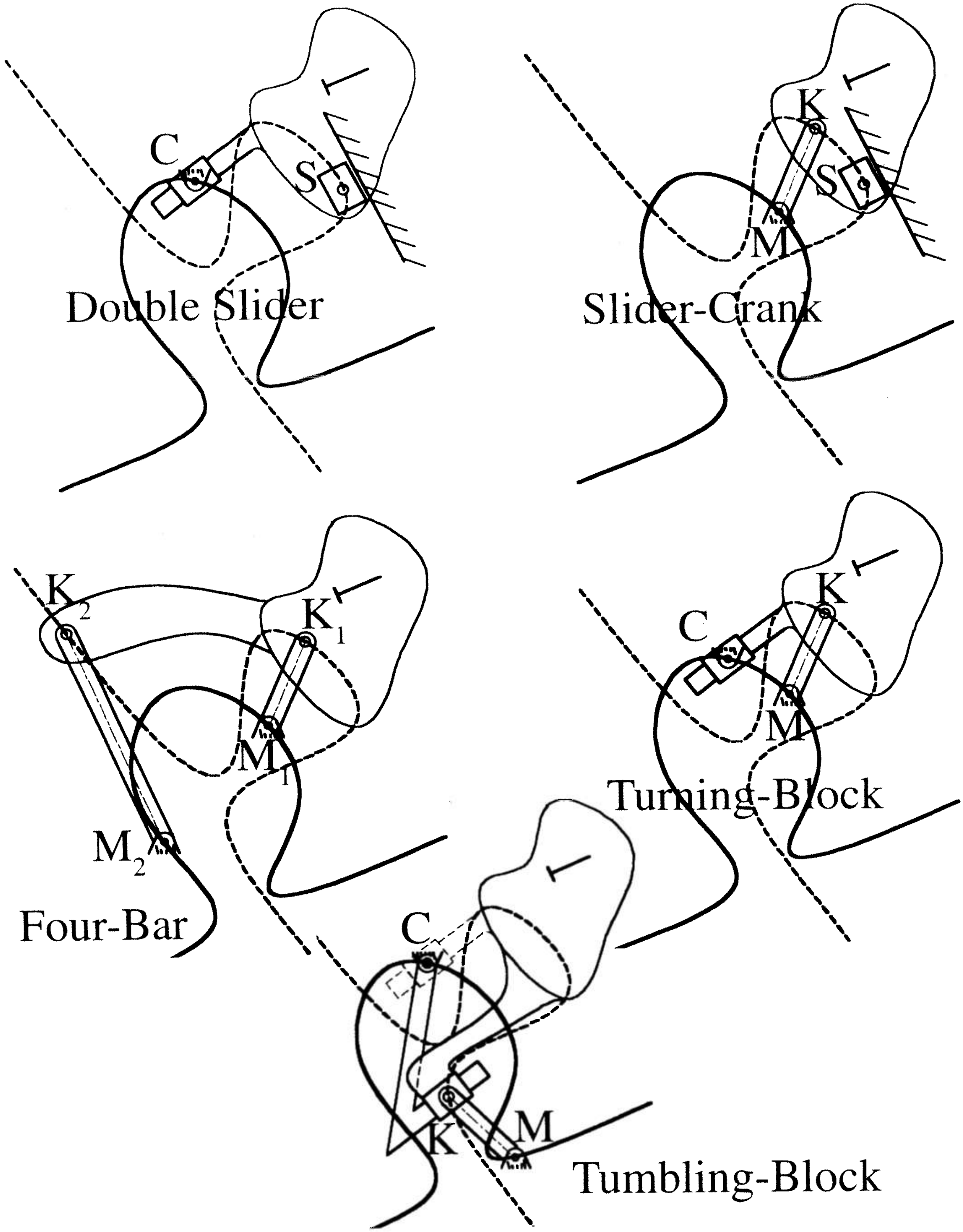
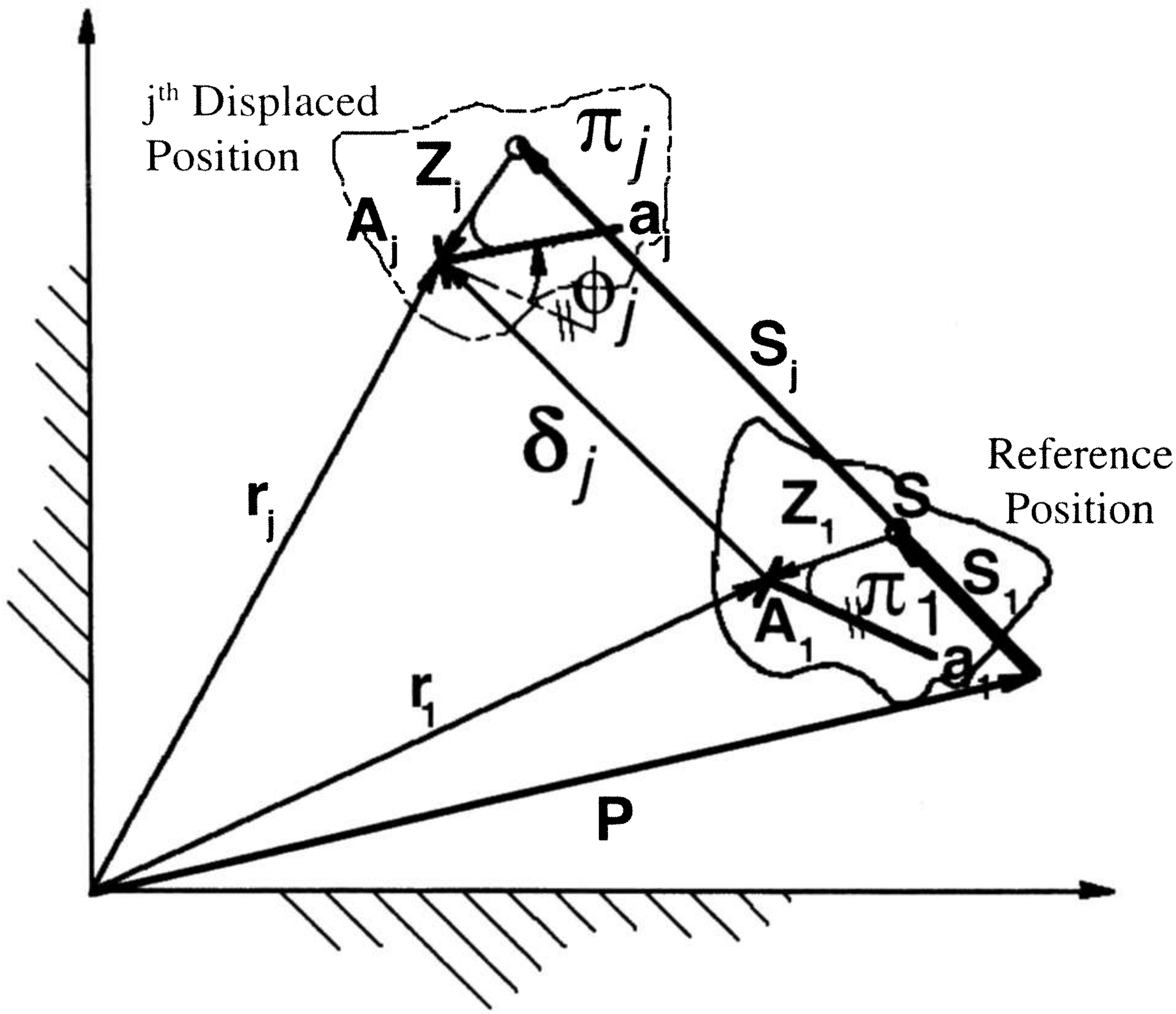
This is a special, limited one time offer. Offer is void where prohibited. 5 cent rebate in MA, VA, DC. Offer expires January 1, 1976 or with expiration of Dr. K, whichever comes sooner. Read fine legal print for exceptions to offer.

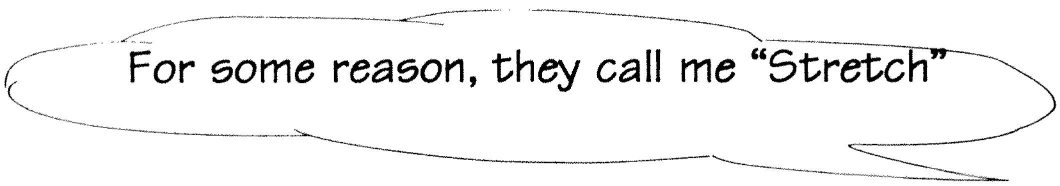

C
S
Double Slider
K
S
M
Slider-Crank
K
2
K
1
M
1
M
2
Four-Bar
C
K
M
Turning-Block
C
K
M
Tumbling-Block

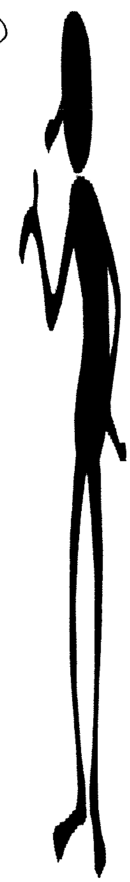
Slider Point Synthesis



Let **S** be the *unknown* sliderpoint in it's starting position. With respect to the origin, point S is located in it's reference posi tion by the sum of the unknown vector **P** (which points to an arbitrary point on the line of sliding) and vector $\mathbf{S_1}$ which is aimed along the line of sliding. When the body moves to a typical jth displaced position, the sliderpoint is located by $\mathbf{P} + \mathbf{S_j}$.

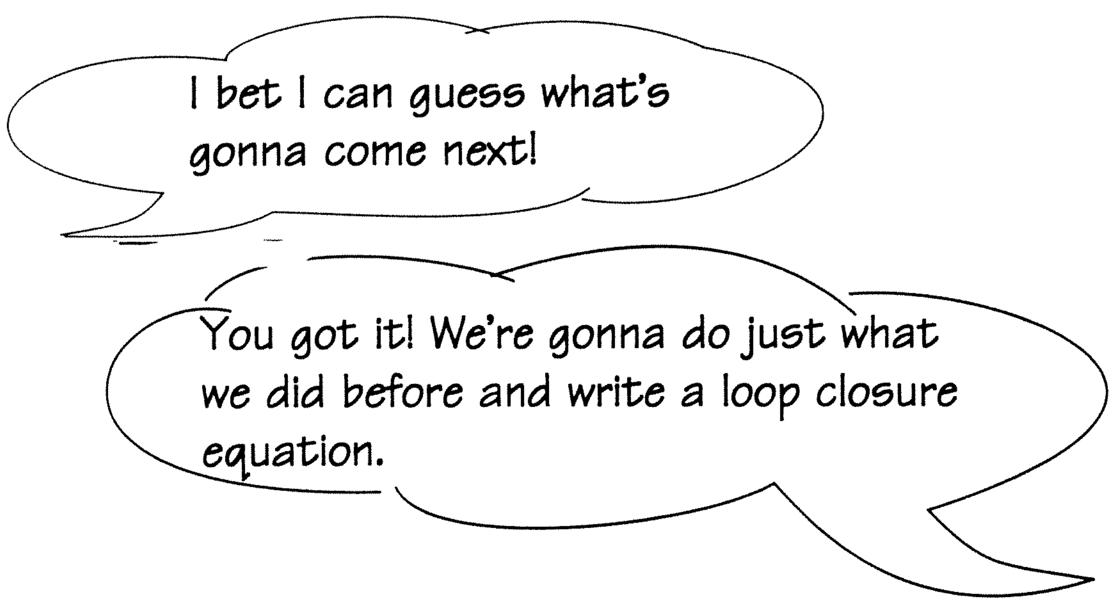
$\mathbf{S_j}$ can be expressed as $\lambda_j \mathbf{S_1}$ where λ_j can be thought of as being a scalar "stretch" ratio.

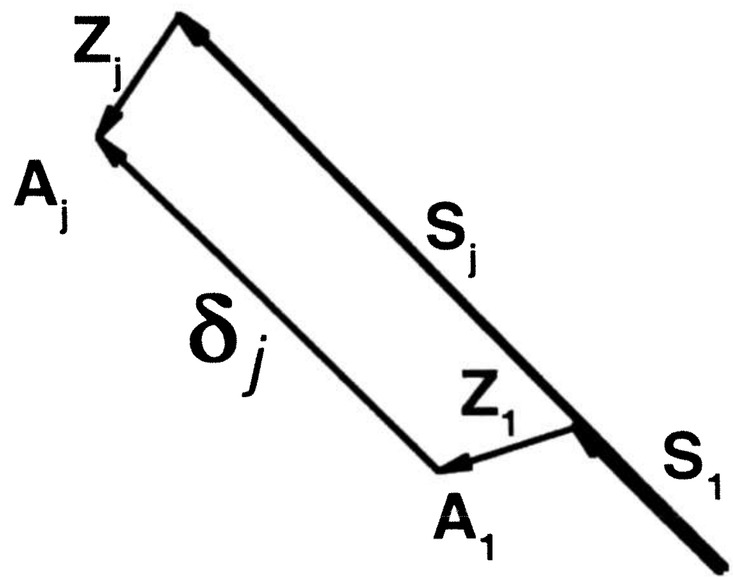




As before, the unknown vector $\mathbf{Z_1}$ is rigidly attached to the moving body in its first position and shows where the given reference point $\mathbf{A_1}$ is located relative to the slider point.

Vector $\mathbf{Z}$ is rigidly attached to the moving body and goes to position $\mathbf{Z_j}$ when the moving body has translated and rotated to the jth design position.





Closure of the above vector polygon is expressed by

$$\delta_j = (e^{i\phi_j} - 1)\, \mathbf{Z_1} + (\lambda_j - 1)\, \mathbf{S_1}$$

$$j = 2, 3, \ldots n$$

Sliderpoint for Three Finitely Separated Positions

For three positions we have the following system of equations:

$$\begin{bmatrix} (e^{i\phi_2} - 1) & (\lambda_2 - 1) \\ (e^{i\phi_3} - 1) & (\lambda_3 - 1) \end{bmatrix} \begin{bmatrix} \mathbf{Z_1} \\ \mathbf{S_1} \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix}$$

For arbitrarily chosen values of the scalars, λ_2 and λ_3 (provided that $\lambda_j \neq 1$) we can solve for the location of the sliderpoint S. Holding λ_3 fixed, say $\lambda_3 = 2$, and varying λ_2 we obtain a circular locus of sliderpoints in the moving body, together with their associated directions of travel in the fixed body.

Making the substitutions $\Lambda_2 = \lambda_2$ - 1 and $\Lambda_3 = \lambda_3$ - 1 and then solving for **Z** and **S** yields:

$$\mathbf{Z}_1 = \frac{\delta_2\Lambda_3 - \delta_3\Lambda_2}{(e^{i\phi_2} - 1)\Lambda_3 - (e^{i\phi_3} - 1)\Lambda_2}$$

$$\mathbf{S}_1 = \frac{(e^{i\phi_2} - 1)\,\delta_3 - (e^{i\phi_3} - 1)\,\delta_2}{(e^{i\phi_2} - 1)\Lambda_3 - (e^{i\phi_3} - 1)\Lambda_2}$$

In this form, the equation for **Z** may be seen to be a form of bilinear mapping or a linear fractional transformation which maps points on the Λ_2 (or Λ_3) line into points on the sliderpoint circle. Simultaneously varying Λ_2 and Λ_3 does *not* yield a two-dimensional surface of possible three-position sliderpoints but merely moves the reference point for the end of **P** in or out along the direction of sliding.

The sliderpoint locus may be shown to be a circle containing the poles P_{12}, $P_{23}^{(1)}$, and P_{31}. Thus, these poles are sliderpoints for three positions.(Yet another example of point-position reduction!)

Further, all sliderpoints slide in the direction of the orthocenter of the pole triangle. Also,... but I digress yet again.

Sliding on to Four Positions

Four positions of a slider yields three complex loop closure equations. For a solution to exist, the augmented coefficient matrix must be singular:

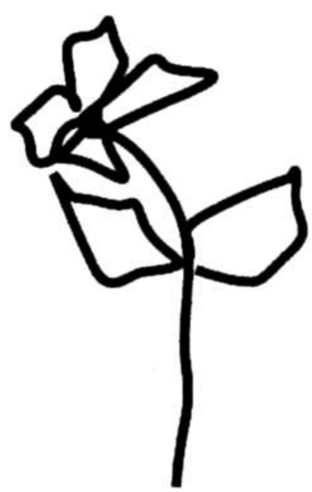
$$\begin{vmatrix} (e^{i\phi_2} - 1) & (\lambda_2 - 1) & \delta_2 \\ (e^{i\phi_3} - 1) & (\lambda_3 - 1) & \delta_3 \\ (e^{i\phi_4} - 1) & (\lambda_4 - 1) & \delta_4 \end{vmatrix} = 0$$

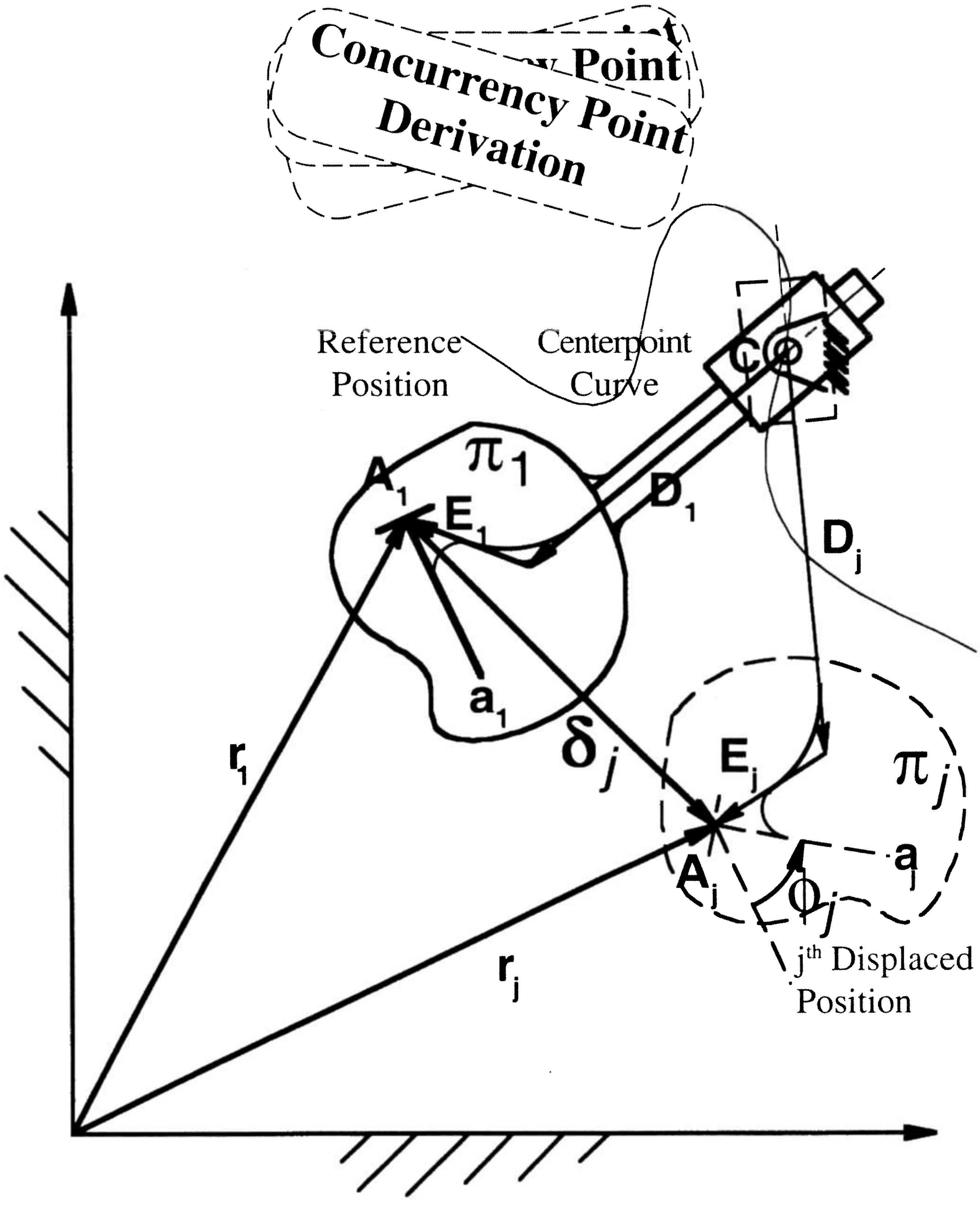
That's your Compatibility Condition.
Deal with it!

Here, one λ say λ_2, can be arbitrarily chosen so long as it doesn't equal 1. Then, expanding in terms of cofactors of the second column we get $\lambda_3\Delta_3 + \lambda_4\Delta_4 = \Delta_1 - \lambda_2\Delta_2$ where the Δ_j's arethe same as they were before in the three position case. We can let $\lambda_2 = 2$, for example, and then solve for a λ_3 and a λ_4 that are compatible with that choice. One way to do this is to separate out the real and imaginary parts of the compatibility equation as

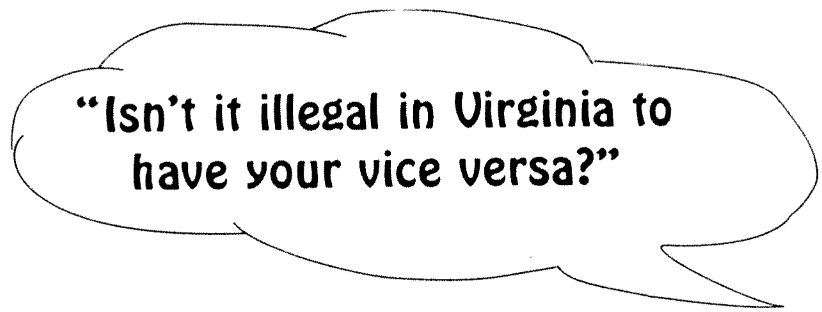
$$\begin{bmatrix} \Delta_{3x} & \Delta_{4x} \\ \Delta_{3y} & \Delta_{4y} \end{bmatrix} \begin{bmatrix} \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} \Delta_{1x} & -\lambda_2\Delta_{2x} \\ \Delta_{1y} & -\lambda_2\Delta_{2y} \end{bmatrix}$$

and then solve this system of two real equations for the two real scalar unknowns λ_3 and a λ_4. Then λ_3, λ_3, and a λ_4 will form a compatible set of stretch factors and we can solve any two of the original system of equations for **Z** and **S**. Notice that varying the choice of λ_2, does not change the slider point in any way, shape, or form but merely changes the reference point along the line of sliding to which the vector **P** points. It doesn't alter the sliderpoint which is obtained. The slidepoint is the sliderpoint is the sliderpoint just like a rose is a rose is a rose and there is only one possible sliderpoint for a general four position motion. Now that I think about it, that's really dumb, since a rose may be a rose but there are lots of different kinds of roses so why do people say a rose is a rose is a rose anyway?



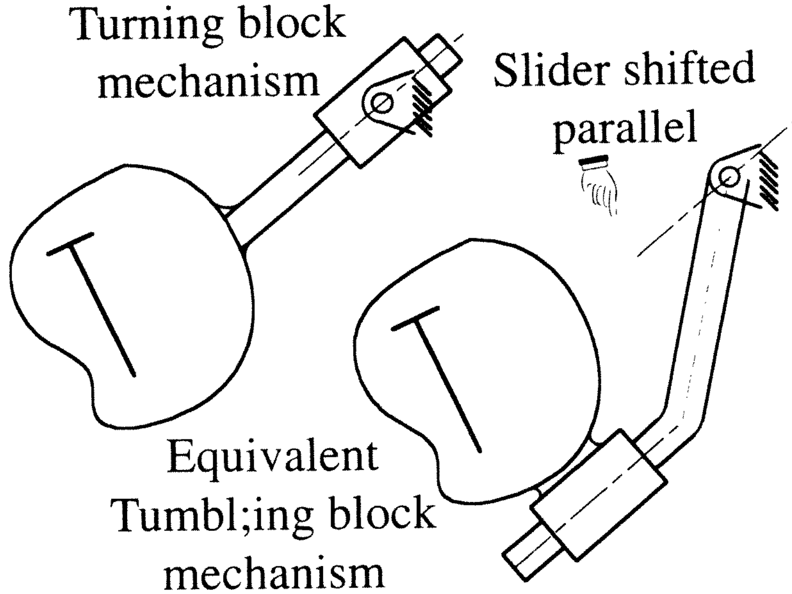

Concurrency Point
Derivation
Reference
Position
Centerpoint
Curve
C
π_1
A_1
E_1
D_1
D_j
a_1
δ_j
E_j
π_j
r_1
A_j
a_j
ϕ_j
r_j
j^{th} Displaced
Position

In this figure, **C** represents the unknown concurrency point we are looking for in the fixed body. It lies somewhere on the centerpoint curve, with a corresponding circlepoint at infinity! In other words, there is some as yet undetermined straight line fastened rigidly to the moving body which [passes through C in all positions of the body. The sliderpoint and the concurrency point are related by a kinematic inversion. The sliderpoint of the motion is the concurrency point of the inverted motion and vice versa.



Let **D** be a vector from point **C** along the straight guide fastened to the moving body.

Thus, point **C** can be used as a turning slideblock for guidance of the body. Alternatively, notice that the same relative motion is obtained by having a straight link at **C** and moving the slide along vector **D** (or parallel to vector **D**) to any other point of the body. Thus, once point **C** has been determined, it can be used for the design of any tumbling block mechanism for four position body guidance!

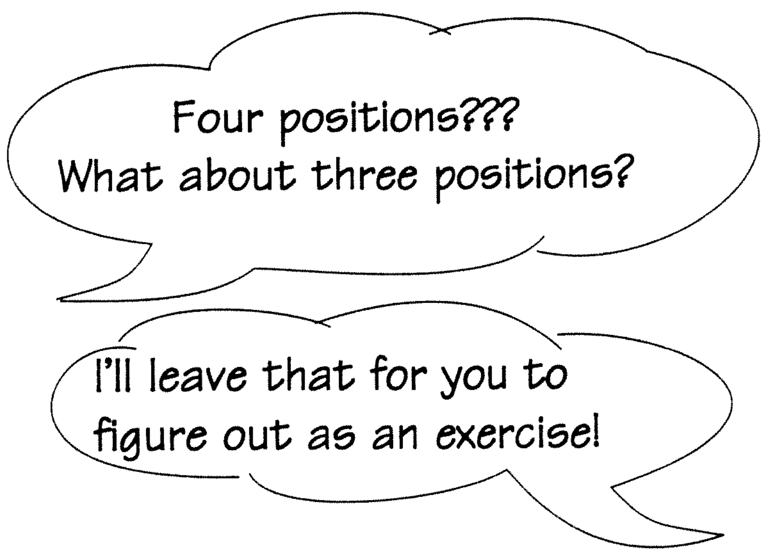


Now, to solve for point **C**, let **E** be an unknown vector rigidly fastened to the moving plane. **E** locates the arbitrarily chosen reference point **A** with respect to some point on the line of sliding in the moving frame. **D** is an unknown vector in the moving system which points along the line of sliding to the end of vector **E**. The tail of vector **D** locates the concurrency point.

As the moving plane π goes from π_1 to π_j, vectors **D** and E rotate with it through the specified angle ϕ_j. At the same time, vector **D** changes length due to sliding of the body through the concurrency point.

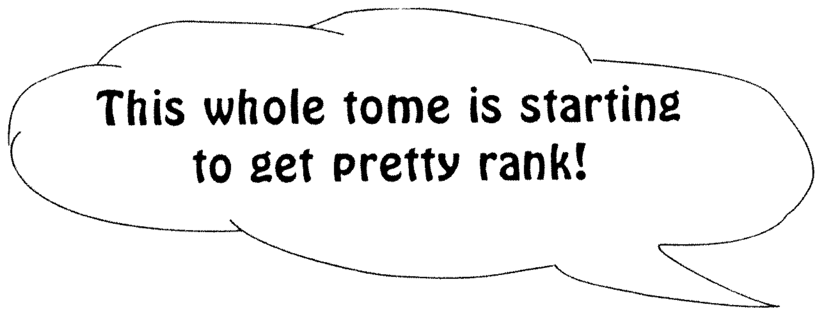
Let $\rho_j = |\mathbf{D}_j| \, / \, |\mathbf{D}_1|$ represent the scalar " stretch" in **D**.

Then, for four finitely separated positions, the following loop closure equations can be written:



$$\begin{bmatrix} (e^{i\phi_2} - 1) & \rho_2(e^{i\phi_2} - 1 \\ (e^{i\phi_3} - 1) & \rho_3(e^{i\phi_3} - 1 \\ (e^{i\phi_4} - 1) & \rho_4(e^{i\phi_4} - 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{D}_1 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

As before, this system of three non-homogeneous complex equations in E and D can only have a solution if the augmented coefficient matrix is of rank 2:



$$\begin{vmatrix} (e^{i\phi_2} - 1) & \rho_2(e^{i\phi_2} - 1 & \delta_2 \\ (e^{i\phi_3} - 1) & \rho_3(e^{i\phi_3} - 1 & \delta_3 \\ (e^{i\phi_4} - 1) & \rho_4(e^{i\phi_4} - 1 & \delta_4 \end{vmatrix} = 0$$

Consistency Condition.

Let $\Lambda_j = e^{i\phi_j} \Delta_j$

Here, the $\mathbf{D}_j$ are the very same 2 by 2 cofactors we used before. With this notation, the compatibility condition becomes

$$\rho_3\Lambda_3 + \rho_4\Lambda_4 = \Delta_1 - \rho_2\Lambda_2$$

after you have expanded the compatibility determinant in terms of cofactors of the second column.

To solve this, one stretch factor, say ρ_2, can be arbitrarily chosen as long as it isn't chosen to be one. For instance, let $\rho_2 = 2$. Then one can solve as before for r3 and r4.

Just as with the sliderpoint, varying ρ_2 does not change the concurrency point location. It merely changes the point on the line of sliding to which vector **E** is referenced. Thus, for four positions, there is only one concurrency point and (since they are kinematic inversions of one another) one slider point.

For three positions, there is a circular locus of concurrency points. This locus contains $\mathbf{P}_{12}$, $\mathbf{P}_{23}$, and $\mathbf{P}_{32}$ along with the orthocenter of the image pole triangle through which all lines of sliding pass.

And now, to the accompaniment of thunderous applause, comes the moment you have been waiting for:

Five Position Burmester Theory!

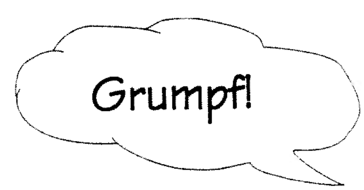
(The mathematical theory behind this stuff is based on the pioneering work of George N. Sandor and Ferdinand Freudenstein, two of the nicest, most brilliant people I have ever known. I threw in a few bits and pieces but they did the hard stuff.)

Ferdinand Freudenstein 1926 - 2006

George Sandor 1911 - 1996

In the 1960s, Professor Ferdinand Freudenstein (known as the "Father of Modern Kinematics"), revolutionized the field of mechanical design by ushering in the computer age in kinematics synthesis and the design of mechanism. George Sandor was his first doctoral student. I was fortunate to be George's first doctoral student and to have Ferdinand serve on my examining committee. At last count, more than five hundred doctoral students spanning five generations have descended from "The Freudenstein Doctoral Tree."

Fascinating though this five position stuff may be for the mathematically inclined, five position Burmester theory is actually a bit of a dud in practice. On rare occassions it may prove useful, but for day in and day out mechanism design there is nothing like two-position theory or maybe three or four position theory if worst comes to worst. Having fewer kinematic constraints means you can satisfy many more real-world mechanism requirements as well as just hitting a bunch of design positions! That is apt to keep your boss happier than just seeing you buried under pages of equations and calculations!



For five finite positions, we have the following system of four complex displacement equations:

$$(e^{i\phi_j} - 1)\,\mathbf{Z}_1 + (e^{i\gamma_j} - 1)\,\mathbf{W}_1 = \delta_j\,,\ j = 2, 3, 4, 5$$

We now have four complex equations, four scalar unknowns ($\gamma_2, \gamma_3, \gamma_4$, and γ_5) and two complex unknowns ($\mathbf{Z}_1$ and $\mathbf{W}_1$). All told, there are eight scalar unknowns and eight scalar equations. If only we knew the values for the gammas we would have a system linear in the $\mathbf{Z}_1$ and $\mathbf{W}_1$ but alas, we don't have that information. We must first laboriously figure out a compatible solution for the gammas.

For a solution to exist, the 4 by 3 matrix

$$\begin{bmatrix} (e^{i\phi_2} - 1) & (e^{i\gamma_2} - 1) & \delta_2 \\ (e^{i\phi_3} - 1) & (e^{i\gamma_3} - 1) & \delta_3 \\ (e^{i\phi_4} - 1) & (e^{i\gamma_4} - 1) & \delta_4 \\ (e^{i\phi_5} - 1) & (e^{i\gamma_5} - 1) & \delta_5 \end{bmatrix}$$

must be of rank 2. A necessary (but not sufficient) condition for this matrix to be of rank 2 is satisfaction of the following two compatibility conditions:

$$\begin{vmatrix} (e^{i\phi_2}-1) & (e^{i\gamma_2}-1) & \delta_2 \\ (e^{i\phi_3}-1) & (e^{i\gamma_3}-1) & \delta_3 \\ (e^{i\phi_4}-1) & (e^{i\gamma_4}-1) & \delta_4 \end{vmatrix} = 0$$

$$\begin{vmatrix} (e^{i\phi_2}-1) & (e^{i\gamma_2}-1) & \delta_2 \\ (e^{i\phi_3}-1) & (e^{i\gamma_3}-1) & \delta_3 \\ (e^{i\phi_5}-1) & (e^{i\gamma_5}-1) & \delta_5 \end{vmatrix} = 0$$

(Note that one could view the five-position problem as two superimposed four-position problems. The first compatibility condition is fulfilled by circlepoints and centerpoints for positions 1, 2, 3, and 4. The second condition covers positions 1, 2, 3, and 5. If both equations are satisfied simultaneously, it would seem that the intersections of the two sets of loci would yield circlepoints (and centerpoints) valid for all five positions. That's true, but they are buried among a bunch of spurious intersections.

Brief parenthetical expression to expand on the above brief parenthetical expression

(I hope your compiler allows multiple levels of nesting)

As seen in the last section, centerpoints for positions 1, 2, 3, and 4 include the poles

$$\mathbf{P}_{12} \quad \mathbf{P}_{13} \quad \mathbf{P}_{14}$$

$$\mathbf{P}_{23} \quad \mathbf{P}_{24}$$

$$\mathbf{P}_{34}$$

and centerpoints for positions 1, 2, 3, and 5 include the poles

$$\mathbf{P}_{12} \quad \mathbf{P}_{13} \quad \mathbf{P}_{15}$$

$$\mathbf{P}_{23} \quad \mathbf{P}_{25}$$

$$\mathbf{P}_{35}$$

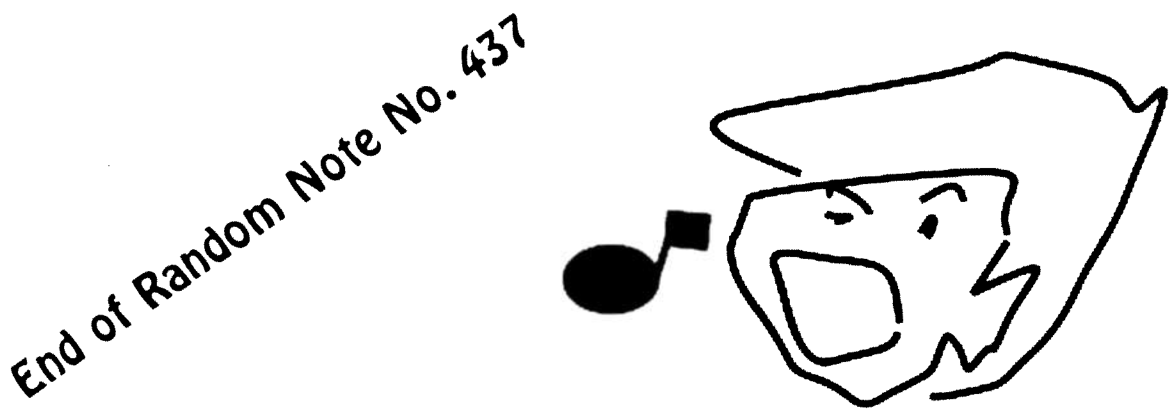
so the common intersections would include the points $\mathbf{P}_{12}$, $\mathbf{P}_{13}$, and $\mathbf{P}_{23}$.

In formulating our five position compatibility equations we had a choice of ten possible pairs of compatibility equations. (There were five groups of four positions possible. We picked two of the five groups. Thus we had five choices for the first compatibility equation and four choices for the second— but it makes no difference which we call first or second.)

Since these common intersections ($\mathbf{P}_{12}$, $\mathbf{P}_{13}$, and $\mathbf{P}_{23}$) are not valid for other compatibility equation groupings, they must be thrown out. Furthermore, the sliderpoint and the concurrency points ($\gamma_j = 0$ and

$\gamma_i = \phi_j$.) differ for the various sets of groupings so they must also be discarded.

One can show that the circlepoint and centerpoint curves are cubics. Since the centerpoint curves for positions 1, 2, 3, and 4 and 1, 2, 3, and 5 are of the third degree they have 3 X 3 or nine intersections. However, we just observed that the poles $\mathbf{P}_{12}$, $\mathbf{P}_{13}$, and $\mathbf{P}_{23}$ must be discounted along with the two imaginary intersections at infinity. Thus, at most, there are only *four* real circlepoint-centerpoint pairs associated with five arbitrarily specified positions. (In fact, some of these intersections may turn out to be imaginary, so there may be four real "Burmester Point Pairs", two real and two imaginary "Burmester Point Pairs", or all four pairs may turn out to be imaginary and there may be no usable solutions.



When we last saw young Dr. Kildare, he was being rushed to emergency to solve the pair of compatibility equations using his mathematical wizardry. We now rejoin our main feature.

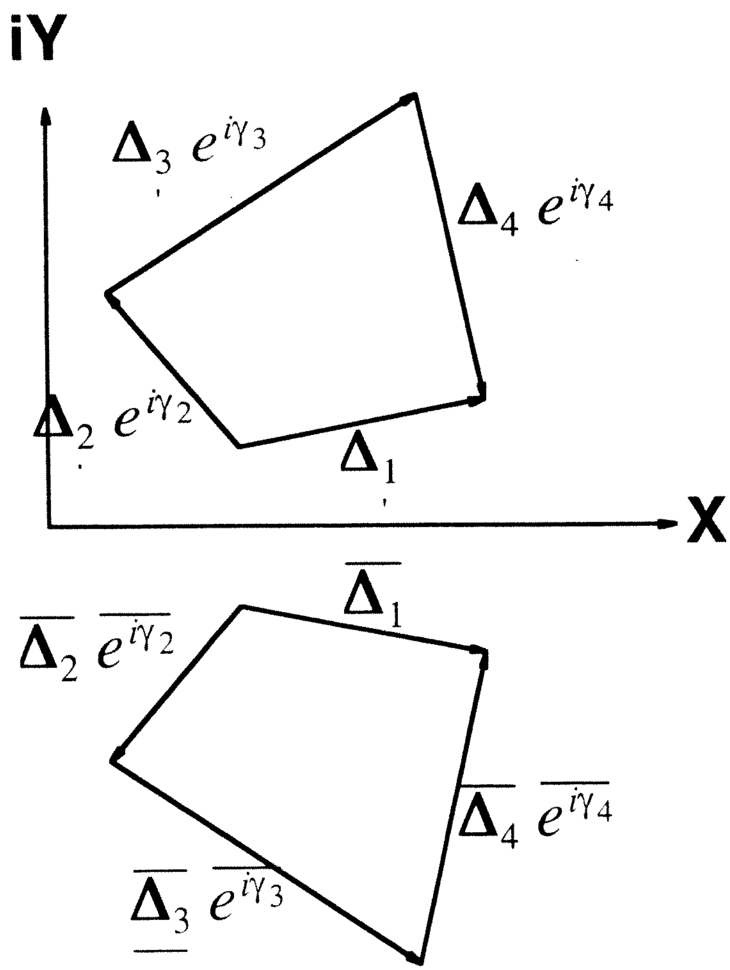
Expand both compatibility equations in terms of the second column. This yields the pair of equations:

$$-\Delta_1 + \Delta_2 e^{i\gamma_2} + \Delta_3 e^{i\gamma_3} + \Delta_4 e^{i\gamma_4} = 0 \qquad (1)$$

$$-\Delta_1' + \Delta_2' e^{i\gamma_2} + \Delta_3' e^{i\gamma_3} + \Delta_4 e^{i\gamma_5} = 0 \qquad (2)$$

where Δ_1, Δ_2, Δ_3, and Δ_4 are as defined before and where the Δ_j' differ from the Δ_j in that they have #5 subscripts in place of the #4 subscripts. (Notice that the cofactor for the bottom element is Δ_4 as before and not Δ_5.)

Equations (1) and (2) are loop closure equations. If we take the complex conjugate of each element in these equations we can create a second set of valid complex equations. (We can picture these new equations as being a mirror image or reflection about the X axis of the original equations. This picture illustrates the complex conjugate of equation (1) for example:



Let's call the complex conjugates of equations (1) and (2) equations (3) and (4). This gives us the following set of four loop closure equations :

$$
\begin{aligned}
-\Delta_1 + \Delta_2\, e^{i\gamma_2} + \Delta_3\, e^{i\gamma_3} + \Delta_4\, e^{i\gamma_4} &= 0 \qquad (1)\\
-\Delta_1' + \Delta_2'\, e^{i\gamma_2} + \Delta_3'\, e^{i\gamma_3} + \Delta_4\, e^{i\gamma_5} &= 0 \qquad (2)\\
-\overline{\Delta_1} + \overline{\Delta_2}\, \overline{e^{i\gamma_2}} + \overline{\Delta_3}\, \overline{e^{i\gamma_3}} + \overline{\Delta_4}\, \overline{e^{i\gamma_4}} &= 0 \qquad (3)\\
-\overline{\Delta_1'} + \overline{\Delta_2'}\, \overline{e^{i\gamma_2}} + \overline{\Delta_3'}\, \overline{e^{i\gamma_3}} + \overline{\Delta_4}\, \overline{e^{i\gamma_5}} &= 0 \qquad (4)
\end{aligned}
$$

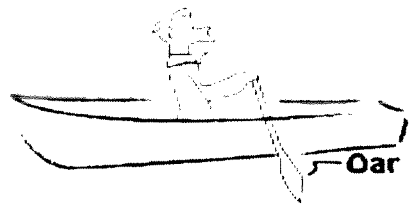
(Once again, notice that $\Delta_4 = \Delta_4'$.)

Now we can eliminate γ_4 from equations (1) and (3) by shuffling the equations around to isolate the γ_4 terms and then multiplying through by the complex conjugates. After we are done, we can take a brief dinner break at a fine restaurant (we deserve it!) and then do the same thing to eliminate γ_5 from equations (2) and (4).

This will leave us well fed and with the following system of equations:

From equations (1) and (3) we get:

$$
\begin{aligned}
\boldsymbol{\Delta}_4\, e^{i\gamma_4}\, \overline{\boldsymbol{\Delta}_4}\, \overline{e^{i\gamma_4}} &= \boldsymbol{\Delta}_4\overline{\boldsymbol{\Delta}_4} \\
&= \boldsymbol{\Delta}_1\overline{\boldsymbol{\Delta}_1} + \boldsymbol{\Delta}_1\overline{\boldsymbol{\Delta}_2}\overline{e^{i\gamma_2}} \\
&\quad + \boldsymbol{\Delta}_1\overline{\boldsymbol{\Delta}_3}\overline{e^{i\gamma_3}} + \boldsymbol{\Delta}_2\overline{\boldsymbol{\Delta}_1}e^{i\gamma_2} \\
&\quad + \boldsymbol{\Delta}_2\overline{\boldsymbol{\Delta}_2} + \boldsymbol{\Delta}_2\overline{\boldsymbol{\Delta}_3}e^{i\gamma_2}\overline{e^{i\gamma_3}} \\
&\quad + \boldsymbol{\Delta}_3\overline{\boldsymbol{\Delta}_1}e^{i\gamma_3} + \boldsymbol{\Delta}_3\overline{\boldsymbol{\Delta}_2}e^{i\gamma_3}\overline{e^{i\gamma_2}} + \boldsymbol{\Delta}_3\overline{\boldsymbol{\Delta}_3}
\end{aligned}
$$

OR 

$$\mathbf{C}_1\, e^{i\gamma_3} + d_1 + \overline{\mathbf{C}_1}\, \overline{e^{i\gamma_3}} = 0 \qquad (5)$$

where

$$\mathbf{C}_1 = \boldsymbol{\Delta}_3(\overline{\boldsymbol{\Delta}_1} + \overline{\boldsymbol{\Delta}_2}\, \overline{e^{i\gamma_2}})$$

and

$$d_1 = \overline{\boldsymbol{\Delta}_1}\boldsymbol{\Delta}_2 e^{i\gamma_2} + \sum_{j=1}^{3} \boldsymbol{\Delta}_j\overline{\boldsymbol{\Delta}_j} - \boldsymbol{\Delta}_4\overline{\boldsymbol{\Delta}_4} + \boldsymbol{\Delta}_1\overline{\boldsymbol{\Delta}_2}\overline{e^{i\gamma_2}}$$

Similarly, from equations (2) and (4) we get

$$\mathbf{C}_2\, e^{i\gamma_3} + d_2 + \overline{\mathbf{C}_2}\, \overline{e^{i\gamma_3}} = 0 \qquad (6)$$

where

$$\mathbf{C}_2 = \boldsymbol{\Delta}'_3(\overline{\boldsymbol{\Delta}'_1} + \overline{\boldsymbol{\Delta}'_2}\, \overline{e^{i\gamma_2}})$$

and

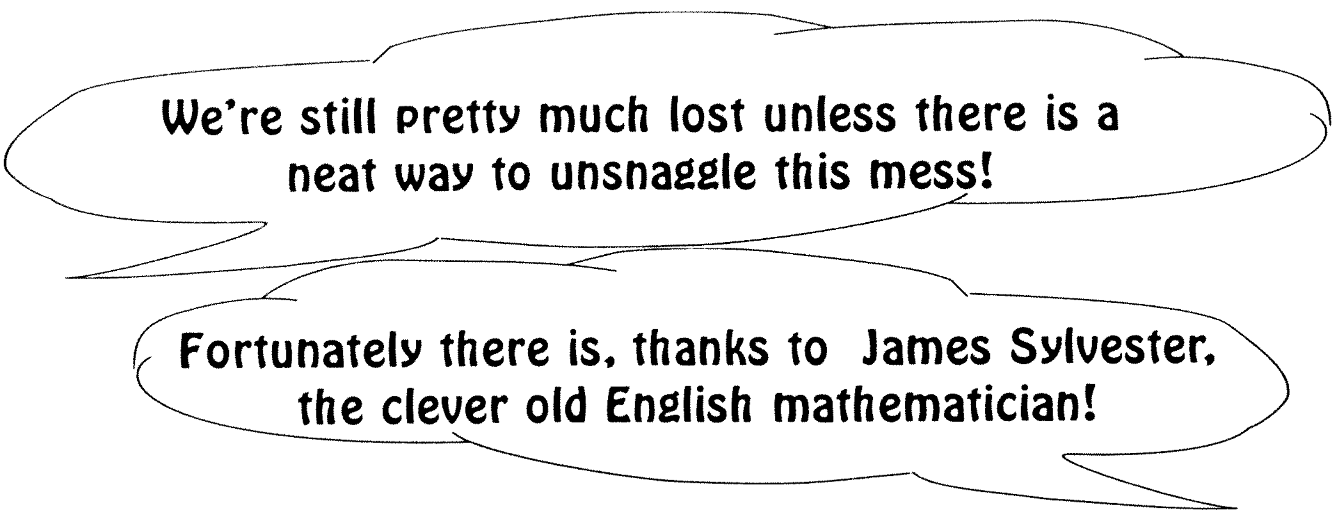
$$d_2 = \overline{\boldsymbol{\Delta}'_1}\boldsymbol{\Delta}'_2 e^{i\gamma_2} + \sum_{j=1}^{3} \boldsymbol{\Delta}'_j\overline{\boldsymbol{\Delta}'_j} - \boldsymbol{\Delta}_4\overline{\boldsymbol{\Delta}_4} + \boldsymbol{\Delta}'_1\overline{\boldsymbol{\Delta}'_2}\overline{e^{i\gamma_2}}$$

Notice that $\mathbf{C}_2$ and d_2 are of the same form as $\mathbf{C}_1$ and d_1 but with primes on the Δ_j for j = 1, 2, and 3. Also, notice that d_1 and d_2 are real numbers while $\mathbf{C}_1$ and $\mathbf{C}_2$ are complex quantities.

Equations (5) and (6) are polynomials in $e^{i\gamma_3}$. We have terms in

$$e^{i2\gamma_3},\ e^{i\gamma_3},\ e^{i0\gamma_3},\ e^{i-\gamma_3}$$

Buried in the coefficients of the polynomial is the unknown, γ_2, along with the known $\mathbf{\Delta}_j$ and $\mathbf{\Delta}_j$' 's.



By multiplying equations (5) and (6) through by $e^{i\gamma_3}$ we can create two more valid equations in powers of $e^{i\gamma_3}$:

$$\mathbf{C}_1\, e^{i2\gamma_3} + d_1\, e^{i\gamma_3} + \overline{\mathbf{C}_1} = 0 \qquad (7)$$

$$\mathbf{C}_2\, e^{i2\gamma_3} + d_2\, e^{i\gamma_3} + \overline{\mathbf{C}_2} = 0 \qquad (8)$$

Consider equations (5), (6), (7), and (8) as being a system of four complex homogeneous simultaneous equations in the unknowns

$$e^{\iota\angle\gamma_3},\ e^{\iota\gamma_3},\ e^{\iota\cup\gamma_3},\ e^{\iota-\gamma_3}$$

For the system to have a solution, the determinant of the coefficients must vanish.

$$\boldsymbol{E} = \begin{vmatrix} 0 & \mathbf{C}_1 & d_1 & \overline{\mathbf{C}_1} \\ 0 & \mathbf{C}_2 & d_2 & \overline{\mathbf{C}_2} \\ \mathbf{C}_1 & d_1 & \overline{\mathbf{C}_1} & 0 \\ \mathbf{C}_2 & d_2 & \overline{\mathbf{C}_2} & 0 \end{vmatrix} = 0 \quad (\textit{Eliminant equation})$$

Vanishing of this eliminant is a *necessary* (but not necessarily sufficient) condition for the equations (5), (6), (7), and (8) to have simultaneous solutions for $e^{\iota\angle\gamma_3},\ e^{\iota\gamma_3},\ e^{\iota\cup\gamma_3},\ e^{\iota-\gamma_3}$

Notice: We now have an equation in γ_2. γ_3 has been eliminated from consideration. This trick for solving equations is known as *Sylvester's Dyalitic Method of Elimination* after the great 19th century English mathematician and lecturer on kinematics, James Sylvester. In 1873 Chebyshev turned Jimmy Sylvester on to kinematics when he wrote Jim

> Yo, Jimboy-
>
> **"Take to kinematics. It will repay you. It is more fecund than geometry. It adds a fourth dimension to space."**

(Actually, historians believe the words "Yo, Jimboy" were accidentally added to Chebyshev's original letter by a careless latter-day scholar who felt the need to bring Chebyshev's slightly archaic wording up to date to appeal to modern day students. That same academician feels that current students might have a significant misunderstanding of the meaning of the word "fecund" as well. Since few students these days own a thesaurus or a dictionary, the author feels it is necessary to explain that "fecund" means "Characterized by great productivity: fertile, fruitful, productive, prolific, rich, giving forth to many possibilities" and has nothing to do with sex in the city.)

Expanding the eliminant in terms of cofactors of the first column gives

$$\mathbf{C}_1 \begin{vmatrix} \mathbf{C}_1 & d_1 & \overline{\mathbf{C}_1} \\ \mathbf{C}_2 & d_2 & \overline{\mathbf{C}_2} \\ d_2 & \overline{\mathbf{C}_2} & 0 \end{vmatrix} - \mathbf{C}_2 \begin{vmatrix} \mathbf{C}_1 & d_1 & \overline{\mathbf{C}_1} \\ \mathbf{C}_2 & d_2 & \overline{\mathbf{C}_2} \\ d_1 & \overline{\mathbf{C}_1} & 0 \end{vmatrix}$$

or

$$d_1 d_2 \mathbf{C}_1 \overline{\mathbf{C}_2} - d_2^2 \mathbf{C}_1 \overline{\mathbf{C}_1} - \mathbf{C}_1 \overline{\mathbf{C}_2}(\mathbf{C}_1 \overline{\mathbf{C}_2} - \mathbf{C}_2 \overline{\mathbf{C}_1})$$
$$+ d_1 d_2 \overline{\mathbf{C}_1} \mathbf{C}_2 - d_1^2 \mathbf{C}_2 \overline{\mathbf{C}_2} + \overline{\mathbf{C}_1} \mathbf{C}_2(\mathbf{C}_1 \overline{\mathbf{C}_2} - \mathbf{C}_2 \overline{\mathbf{C}_1}) = 0$$

Again, by way of reminder,

$$\mathbf{C}_1 = \boldsymbol{\Delta}_3(\overline{\boldsymbol{\Delta}_1} + \overline{\boldsymbol{\Delta}_2}\, \overline{e^{i\gamma_2}})$$

and

$$d_1 = \overline{\boldsymbol{\Delta}_1} \boldsymbol{\Delta}_2 e^{i\gamma_2} + \sum_{j=1}^{3} \boldsymbol{\Delta}_j \overline{\boldsymbol{\Delta}_j} - \boldsymbol{\Delta}_4 \overline{\boldsymbol{\Delta}_4} + \boldsymbol{\Delta}_1 \overline{\boldsymbol{\Delta}_2}\, \overline{e^{i\gamma_2}}$$

$$\mathbf{C}_2 = \boldsymbol{\Delta}_3'(\overline{\boldsymbol{\Delta}_1'} + \overline{\boldsymbol{\Delta}_2'}\, \overline{e^{i\gamma_2}})$$

and

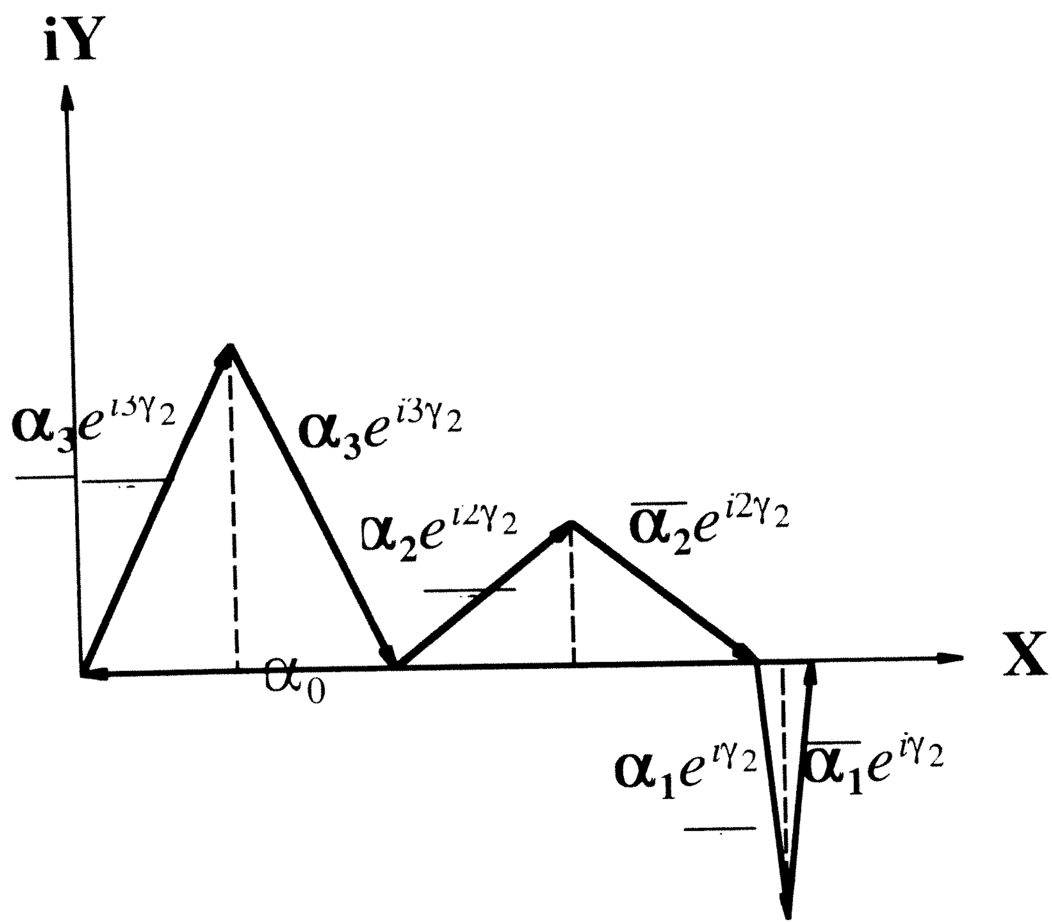
$$d_2 = \overline{\boldsymbol{\Delta}_1'} \boldsymbol{\Delta}_2' e^{i\gamma_2} + \sum_{j=1}^{3} \boldsymbol{\Delta}_j' \overline{\boldsymbol{\Delta}_j'} - \boldsymbol{\Delta}_4 \overline{\boldsymbol{\Delta}_4} + \boldsymbol{\Delta}_1' \overline{\boldsymbol{\Delta}_2'}\, \overline{e^{i\gamma_2}}$$

Thus, the eliminant turns out to be a real polynomial in $e^{i\gamma_2}$. Each term of the eliminant is accompanied by its complex conjugate. **E** is of the form

$$\alpha_3 e^{i3\gamma_2} + \alpha_2 e^{i2\gamma_2} + \alpha_1 e^{i\gamma_2} + \alpha_0 +$$
$$\overline{\alpha_3}\,\overline{e^{i3\gamma_2}} + \overline{\alpha_2}\,\overline{e^{i2\gamma_2}} + \overline{\alpha_1}\,\overline{e^{i\gamma_2}} = 0$$

Here, the α_0 is real and the other α_j's are readily computed functions of the known $\mathbf{\Delta}_j$'s and $\mathbf{\Delta}_j$''s (which in turn were calculated on the basis of the known input $\mathbf{r}_j$'s and ϕ_j's).

We can visualize the eliminant equation as being a vector loop closure equation in which we have a bunch of known α vectors that need to be rotated by different multiples of the unknown angle γ_2 in order to sum to zero:



Since the eliminant is real, we can work with the real portion alone. The real part of the eliminant is:

$$2\begin{pmatrix} \boldsymbol{\alpha_{3x}} \boldsymbol{cos}\ 3\gamma_2 - \boldsymbol{\alpha_{3y}} \boldsymbol{sin}\ 3\gamma_2 + \boldsymbol{\alpha_{2x}} \boldsymbol{cos}\ 2\gamma_2 \\ - \boldsymbol{\alpha_{2y}} \boldsymbol{sin}\ 2\gamma_2 + \boldsymbol{\alpha_{1x}} \boldsymbol{cos}\ \gamma_2 - \boldsymbol{\alpha_{1y}} \boldsymbol{sin}\ \gamma_2 + \boldsymbol{\alpha_0} \end{pmatrix} = 0$$

"The horrendous mess"

We need to solve this horrendous mess for the unknown γ_2. By means of trig identities (you did take trig in high school, didn't you?) we can express $cos(n\gamma_2)$ and $sin(n\gamma_2)$ in terms of one and only one trig function, namely $tan(\frac{n\gamma_2}{2})$!

Among other admirable qualities, the tangent of the half angle is nice because it is single-valued in both its direct and inverse form.

So, plowing ahead towards a solution to this mess, we can make the following substitutions:

$$Let\ \tau = tan(\frac{\gamma_2}{2})$$

Then

$$cos\ \gamma_2 = \frac{1 - \tau^2}{1 + \tau^2}$$

$$sin\ \gamma_2 = \frac{2\ \tau}{1 + \tau^2}$$

$$\cos 2\gamma_2 = \frac{1 - 6\tau^2 + \tau^4}{\left(1 + \tau^2\right)^2}$$

$$\sin 2\gamma_2 = \frac{4\tau(1 - \tau^2)}{\left(1 + \tau^2\right)^2}$$

$$\cos 3\gamma_2 = \frac{\left(1 - \tau^2\right)\left(1 - 14\tau^2 + \tau^4\right)}{\left(1 + \tau^2\right)^3}$$

$$\sin 3\gamma_2 = \frac{2\tau\left(3 - 10\tau^2 + 3\tau^4\right)}{\left(1 + \tau^2\right)^3}$$

With these substitutions and after considerable cursing, erasing, and fiddling with algebra, we massage the *"Horrendous Mess"* equation into the following more attractive and tractable form:

$$0 = \tau^6 + \lambda_5\tau^5 + \lambda_4\tau^4 + \lambda_3\tau^3 + \lambda_2\tau^2 + \lambda_1\tau + \lambda_0$$

In this new form, $\lambda_j = Q_j/Q_6$ for j = 0, 1, 2, 3, 4, and 5 and where

$$Q_6 = -2\,\alpha_{3_x} + 2\alpha_{2_x} - 2\alpha_{1_x} + \alpha_0$$
$$Q_5 = -12\,\alpha_{3_y} + 8\alpha_{2_y} - 4\alpha_{1_y}$$
$$Q_4 = 30\,\alpha_{3_x} - 10\,\alpha_{2_x} - 2\alpha_{1_x} + 3\,\alpha_0$$
$$Q_3 = 40\,\alpha_{3_y} - 8\alpha_{1_y}$$
$$Q_2 = -30\,\alpha_{3_x} - 10\,\alpha_{2_x} + 2\alpha_{1_x} + 3\,\alpha_0$$
$$Q_1 = -12\,\alpha_{3_y} - 8\,\alpha_{2_y} - 4\,\alpha_{1_y}$$
$$Q_0 = 0 = 2\,\alpha_{3_x} + 2\,\alpha_{2_x} + 2\alpha_{1_x} + \alpha_0$$

Now many pages ago when you were but a little tot, naive in what you were getting into and eager for knowledge, you learned that $\gamma_2 = 0$ was a trivial solution to the compatibility equations as was $\gamma_2 = \phi_2$. Therefor,

$$\tau = 0$$
$$\tau = tan\left(\frac{\phi_2}{2}\right)$$

must be solutions to this equation. Using synthetic division, we can reduce this equation from a sextic to a quartic without needing to call in the public health authorities but by merely dividing out these two known roots. We have:

$$0 = \frac{\tau^6 + \lambda_5\tau^5 + \lambda_4\tau^4 + \lambda_3\tau^3 + \lambda_2\tau^2 + \lambda_1\tau}{\tau(\tau - \tau_0)}$$

$$\tau_0 = tan\left(\frac{\phi_2}{2}\right)$$

where

$$\tau_0 = tan\left(\frac{\phi_2}{2}\right)$$

Reshuffling this symbolically gives:

$$0 = \tau^4 + \mu_4\tau^3 + \mu_3\tau^2 + \mu_2\tau + \mu_1$$

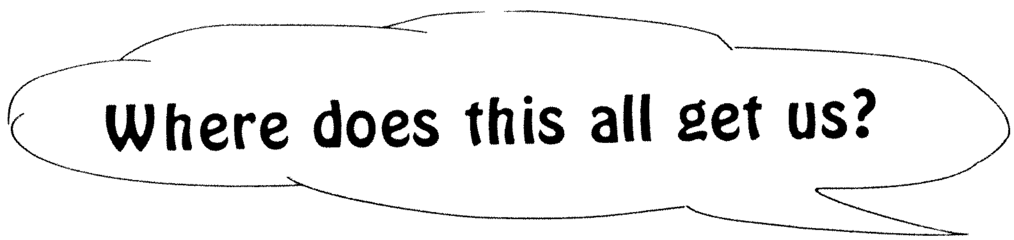
where

$$\mu_4 = \lambda_5 + \tau_0$$

$$\mu_3 = \lambda_4 + \tau_0\lambda_5 + \tau_0^2$$

$$\mu_2 = \lambda_3 + \tau_0\lambda_4 + \tau_0^2\lambda_5 + \tau_0^3$$

$$\mu_1 = \lambda_2 + \tau_0\lambda_3 + \tau_0^2\lambda_4 + \tau_0^3\lambda_5 + \tau_0^4$$



I'm glad you asked. After all this futzing around, we have reduced the five position compatibility problem to the solution of a quartic equation with known, deterministic coefficients. The nice thing about having a quartic is that quartics can be solved in closed-form to yield four, two, or zero real roots, τ_k. Now possibly getting zero real roots is a bummer (after all that work) but it is much better to *know* that there aren't any answers to your question than to spend weeks looking for a solution that you can prove theoretically doesn't exist!

With that in mind, we can plow ahead and solve this quartic in closed-form for up to a maximum of four τ_k's.

For each real root τ_k that we obtain, we can solve for a corresponding

$$\gamma_{2_k} = 2\, tan^{-1}\left(\tau_k\right) \qquad k = 1, 2, 3, 4 \; (\textit{we hope!})$$

Then we can plug these γ_{2k}'s back in and unwind the whole process. We get:

$$\mathbf{C}_{1k} = \Delta_3\left(\overline{\Delta_1} + \overline{\Delta_2}\, \overline{e^{i\gamma_{2k}}}\right)$$

$$d_{1k} = \overline{\Delta_1}\Delta_2 e^{i\gamma_{2k}} + \sum_{j=1}^{3} \Delta_j\overline{\Delta_j} - \Delta_4\overline{\Delta_4} + \Delta_1\overline{\Delta_2}\,\overline{e^{i\gamma_{2k}}}$$

$$\mathbf{C}_{2k} = \Delta'_3\left(\overline{\Delta'_1} + \overline{\Delta'_2}\, \overline{e^{i\gamma_{2k}}}\right)$$

and

$$d_{2k} = \overline{\Delta'_1}\Delta'_2 e^{i\gamma_{2k}} + \sum_{j=1}^{3} \Delta'_j\overline{\Delta'_j} - \Delta_4\overline{\Delta_4} + \Delta'_1\overline{\Delta'_2}\,\overline{e^{i\gamma_{2k}}}$$

Mushing (mooshing??) these values back into our old equations (5) and (6) up to four times (if we were lucky enough to get four real roots) we get:

$$\mathbf{C}_{1k}\, e^{i\gamma_{3k}} + \overline{\mathbf{C}}_{1k}\, \overline{e^{i\gamma_{3k}}} = -\, d_{1k}$$

$$\mathbf{C}_{2k}\, e^{i\gamma_{3k}} + \overline{\mathbf{C}}_{2k}\, \overline{e^{i\gamma_{3k}}} = -\, d_{2k}$$

Last but not least, (well, not *really* last but close enough to last so that I'll give you some false hope) we can solve this pair of equations using Cramer's rule for

$$e^{i\gamma_{3_k}} = \frac{\begin{vmatrix} -d_{1_k} & \overline{\mathbf{C}}_{1k} \\ -d_{2_k} & \overline{\mathbf{C}}_{2k} \end{vmatrix}}{\begin{vmatrix} \mathbf{C}_{1k} & \overline{\mathbf{C}}_{1k} \\ \mathbf{C}_{2k} & \overline{\mathbf{C}}_{2k} \end{vmatrix}}$$

and

$$e^{i\gamma_{4_k}} = \frac{-\left(\boldsymbol{\Delta}_1 + \boldsymbol{\Delta}_2 e^{i\gamma_{2k}} + \boldsymbol{\Delta}_3 e^{i\gamma_{3k}}\right)}{\boldsymbol{\Delta}_4}$$

and

$$e^{i\gamma_{5_k}} = \frac{-\left(\boldsymbol{\Delta}_1' + \boldsymbol{\Delta}_2' e^{i\gamma_{2k}} + \boldsymbol{\Delta}_3' e^{i\gamma_{3k}}\right)}{\boldsymbol{\Delta}_4}$$

and (last but not least)

$$\begin{bmatrix} (e^{i\phi_2} - 1) & (e^{i\gamma_{2_k}} - 1) \\ (e^{i\phi_3} - 1) & (e^{i\gamma_{3_k}} - 1) \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{1_k} \\ \mathbf{W}_{1_k} \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix}$$

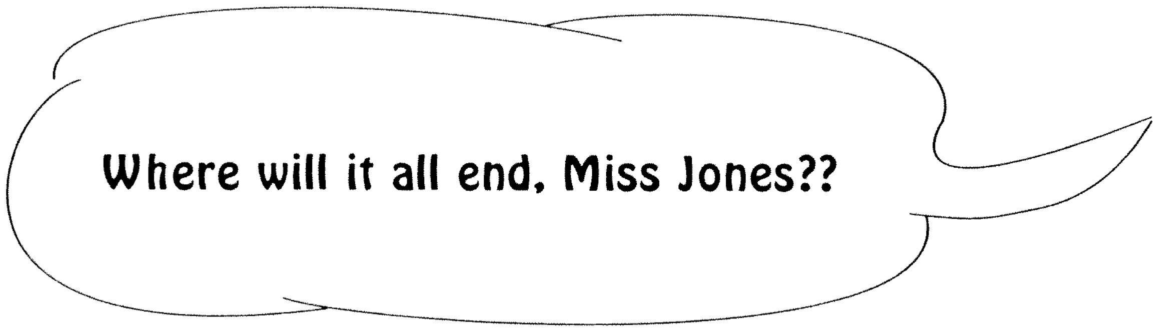
which allows us to solve for

$$\mathbf{Z}_{1_k}, \mathbf{W}_{1_k} \qquad k = 1, 2, 3, 4$$

and, if we are still conscious to appreciate it,

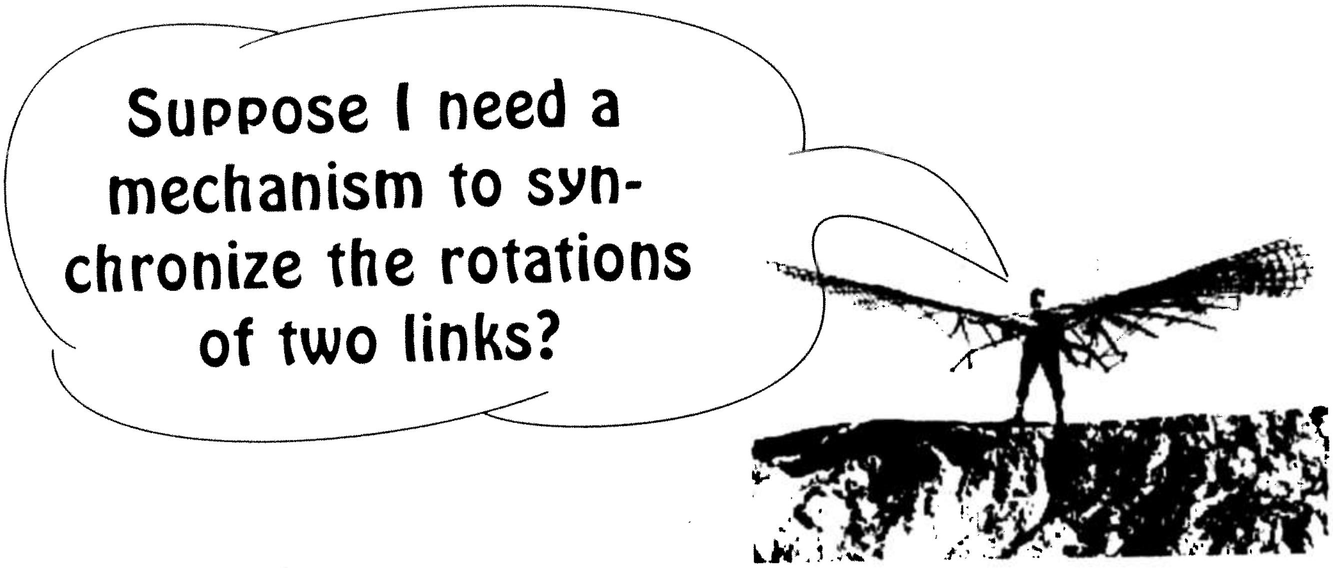
$$\mathbf{K}_{1_k}, \mathbf{M}_{1_k} \qquad k = 1, 2, 3, 4$$

As the proctologist said to his nurse...



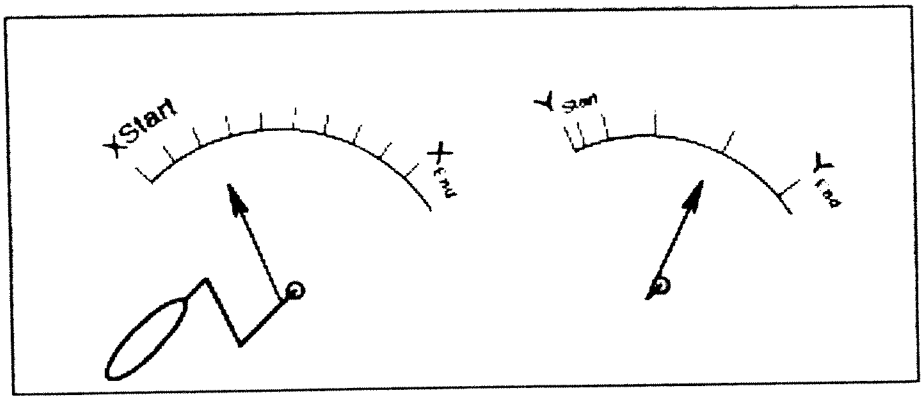
After all that pain and suffering we end up with four, two, or no Burmester Point Pairs from which we can design up to six possible four-bars (in the best of all possible worlds) to move the body through the five given positions. Unfortunately, all the solutions might turn out to be imaginary! Imagine that!

Of course, if the mechanism is to be of any real-world use, it must also have good transmission angles, so our maximum of six possible mechanisms might not include any with good transmission. Also, of course, our mechanism must have the right Grashof type, so of the maximum of six possible mechanisms and of the subset of those that have good transmission angles, there might not be any with the Grashof type we are looking for. Also, of course, the mechanism must meet our space constraints, so of the maximum of six possible mechanisms and of the subset of those that have good transmission angles, and of the subset of those that have the Grashof type we are looking for there still might not be any that meet our space constraints. (Some of the circlepoints or centerpoints might turn out to be in Madagascar wherever the hell that is!) Finally, of the maximum of six possible mechanisms and of the subset of those that have good transmission angles, and of the subset of those that have the Grashof type we are looking for and of the subset of those that meet our space constraints there might still not be any that meet your boss's prejudice against using a linkage mechanism for the job in the first place. Damn. That's life. Ain't it a bitch?



Every now and then you need to design a linkage that will synchronize the rotations of the input and output links in some desired way. For instance, you may be designing a piece of machinery in which a given series of rotations of the motor shaft needs to produce a different series of angular rotations of an output shaft where the relationships are weird kind of like my strange uncle Moishe and the rest of the family.

Kinematicians call this a "Function Generation Problem." (I'm not talking about the problem of my uncle Moishe but I'm talking about the problem of designing a machine like the following:

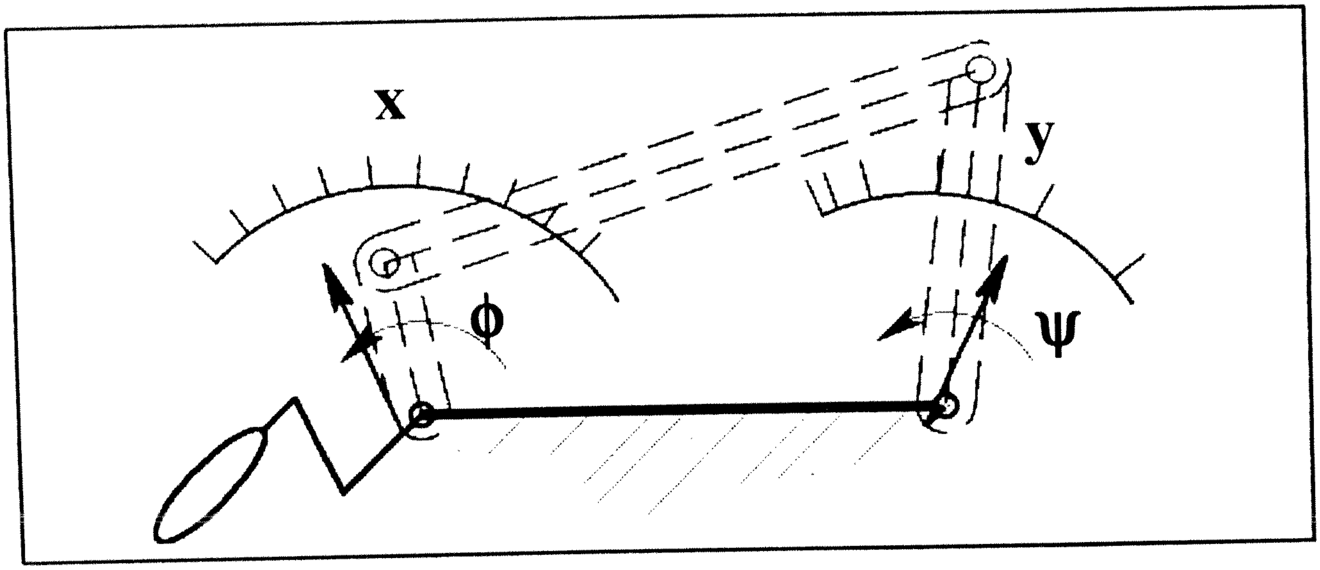


Here, you may have a table giving desired X values versus Y values that need to be matched up at a certain desired number of precision positions, as in:

X	Y
20°	27°
25°	35°
30°	50°
...	...
...	...

With a slight twist on the theory you now know (and a little bit of luck) a four-bar linkage can hopefully be synthesized to sit inside the box and provide the nonlinear coupling between the rotations of the shafts carrying the x and y pointers.

At least for two, three, four, or five design positions...



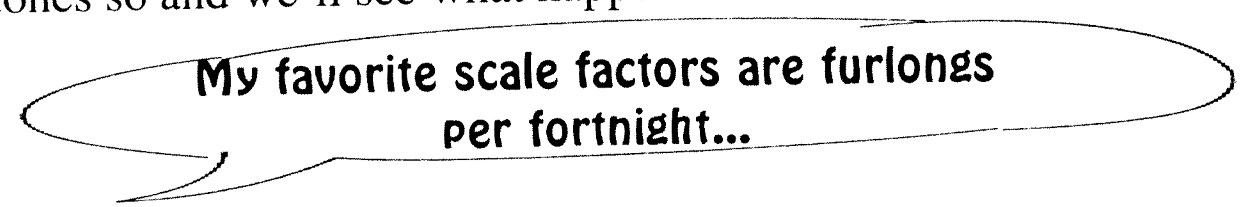
You can let ϕ (the rotations of the input crank) be the linear analog of x and ψ (the rotations of the output crank) be the linear analog of y.

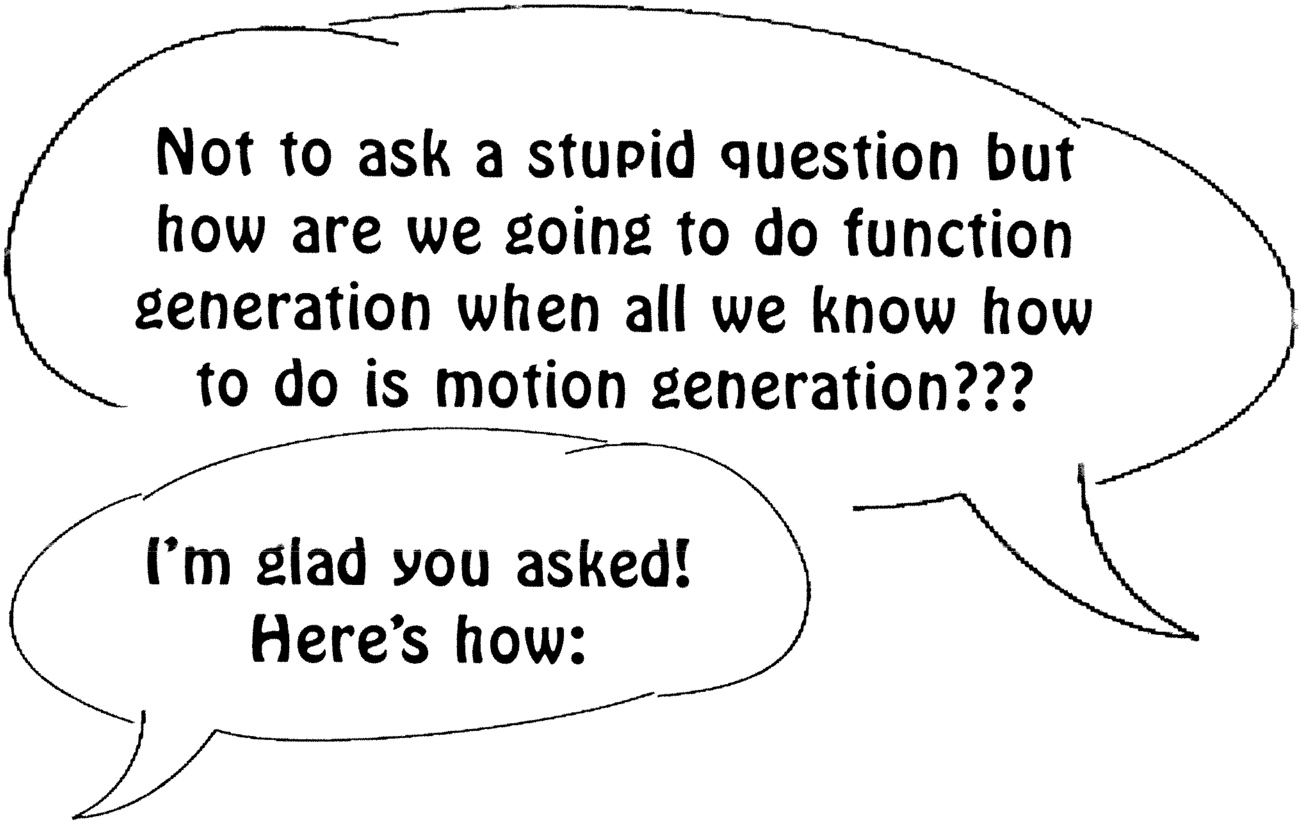
Let

$$\phi_{1j} = R_{\phi}\,(x_j - x_1)$$

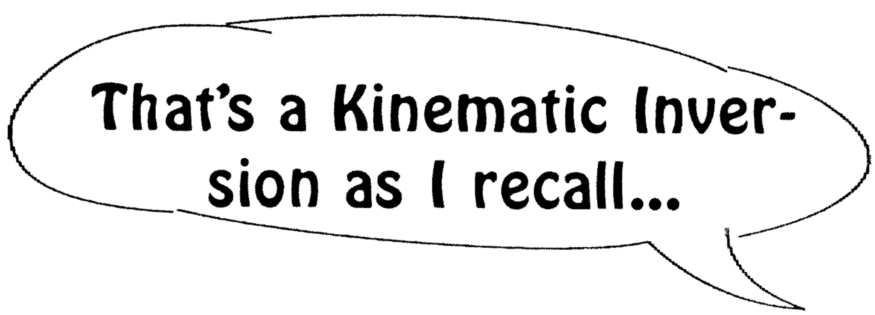
$$\psi_{1j} = R_{\psi}\,(y_j - y_1)$$

Here, $\mathbf{R}_{\phi}$ and $\mathbf{R}_{\psi}$ are scale factors that you can choose so as to give the best performance. You can pick the scale factors using your best judgment, common sense, a Ouiji Board, a dart board, or some fancy mathematical optimization theory but let's assume you have dones so and we'll see what happens next.

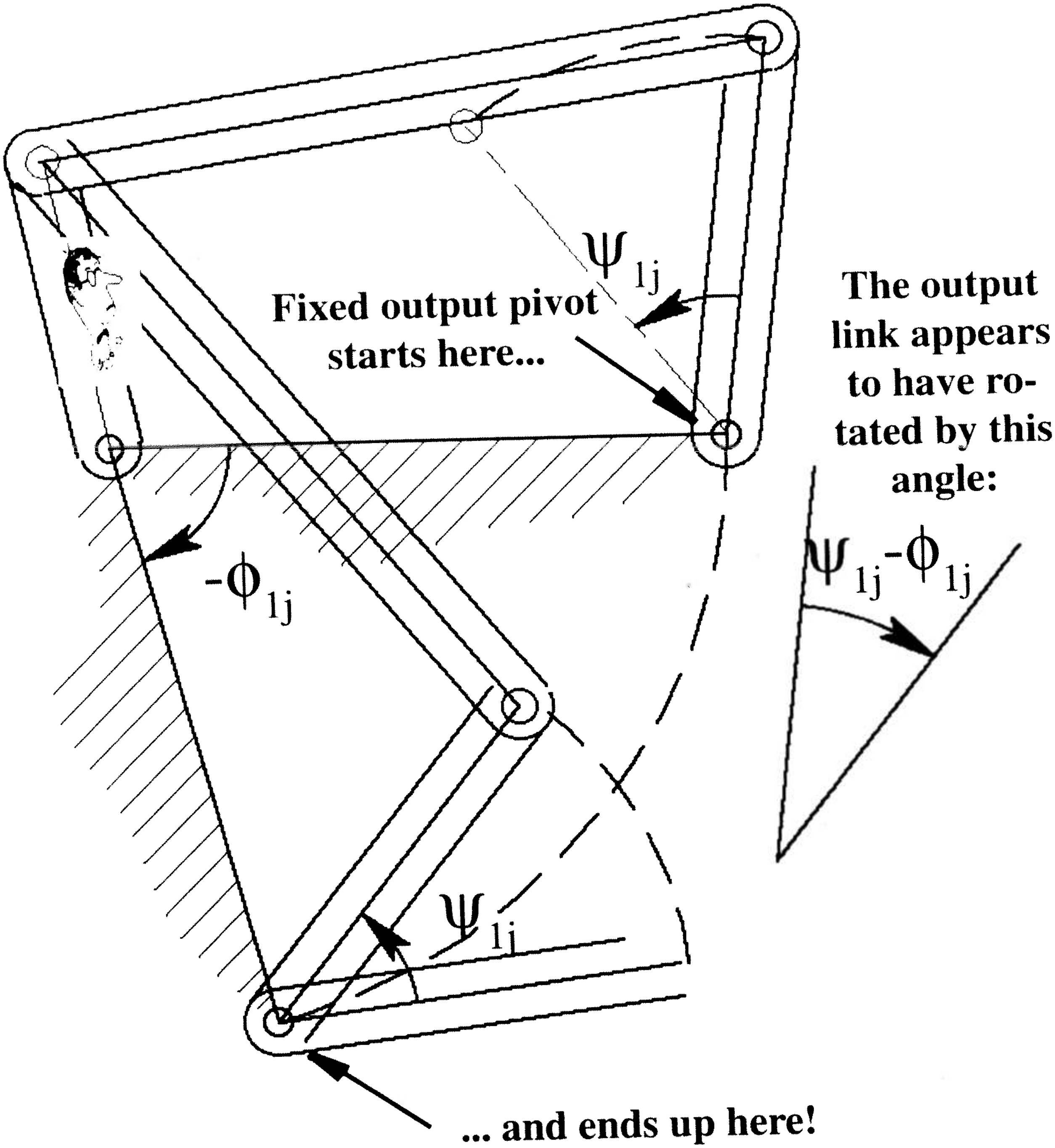




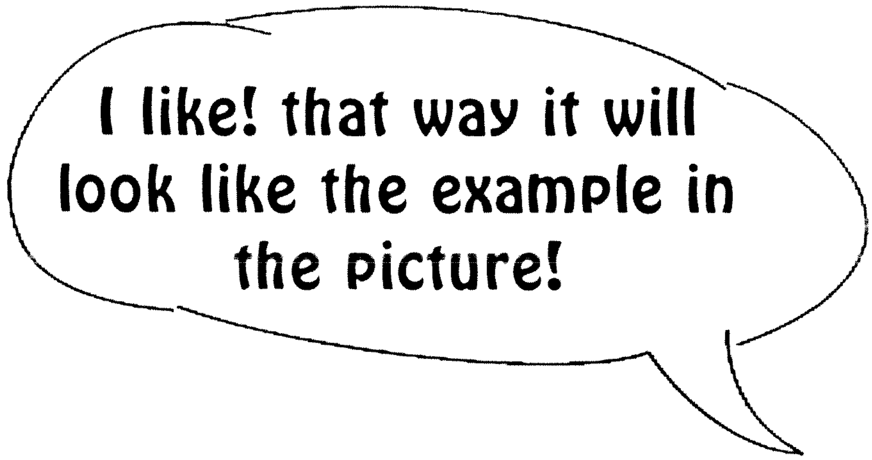
Imagine you are sitting on the input crank watching the linkage move around from its starting #1 (reference) position to a typical j^{th} displaced design position. From your viewpoint, the input crank stays fixed while the frame and everything else moves around



you. If the input crank rotates through an angle ϕ_{1j} while the output crank rotates through ψ_{1j} what would you see happening to the actual frame link and the output fixed pivot? By what relative angle would you see the output crank rotate?



Under the kinematic inversion, you see the opposite fixed pivot swing down away from the link you are riding on through the angle $-\phi_{1j}$. It swings on a circle centered on the input crank's fixed pivot. At the same time, you see the output crank rotating through the angle ψ_{1j}-ϕ_{1j}. Since for function generation (and *only* for function generation) relative proportions are all that are important and not the absolute size or orientation, you can pick the frame link to be a unit horizontal link if you like. Then you can synthesize the function generator linkage using the motion generation techniques you already have mastered! (You have mastered them, haven't you?)



Synthesize the function generator as an inverted motion generator that moves the follower link through its observed *motion* with respect to the input crank link. For instance, if you were doing a five position synthesis (perhaps because you were a glutton for punishment) you would input into your program the following data:

$$\mathbf{r}_j = 1\ e^{-i\phi j}$$

$$\mathbf{\Phi}_j = \psi_{1j} - \phi_{1j}$$

j = 2, 3, 4, 5, etc.

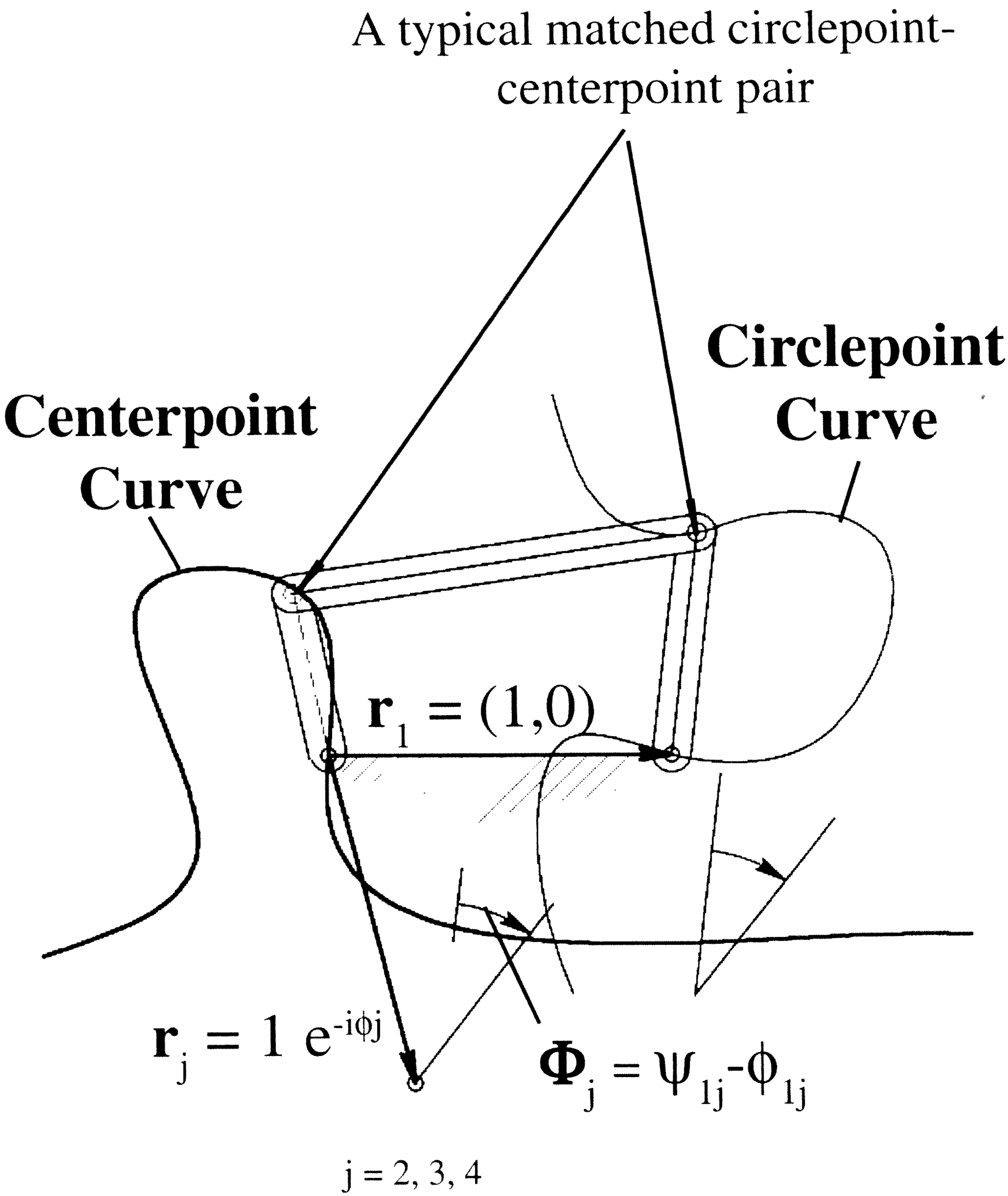
Up to four Burmester Point Pairs might come out of your program if all the solutions are real.

One of the resulting centerpoints will be at (0, 0) and is the fixed pivot beneath the input crank. The corresponding circlepoint will be at (1, 0) and will in reality be the *given* location of the output crank's *fixed* pivot. Each of the remaining circlepoint-centerpoint pairs represents a possible *coupler* link for the function generating four bar. The circlepoint represents the output crank's moving pivot in the reference (#1) design position and the corresponding centerpoint represents the position of the input crank's *moving* pivot in its starting position.

Using five precision positions you get at most three possible function generators for a given data set.

On the other hand, if you are doing a four position synthesis you will get a single infinity of function generators from which to choose. The centerpoint curve will pass through the two pivots on the input crank and the circlepoint curve will pass through the two pivots on the output crank.

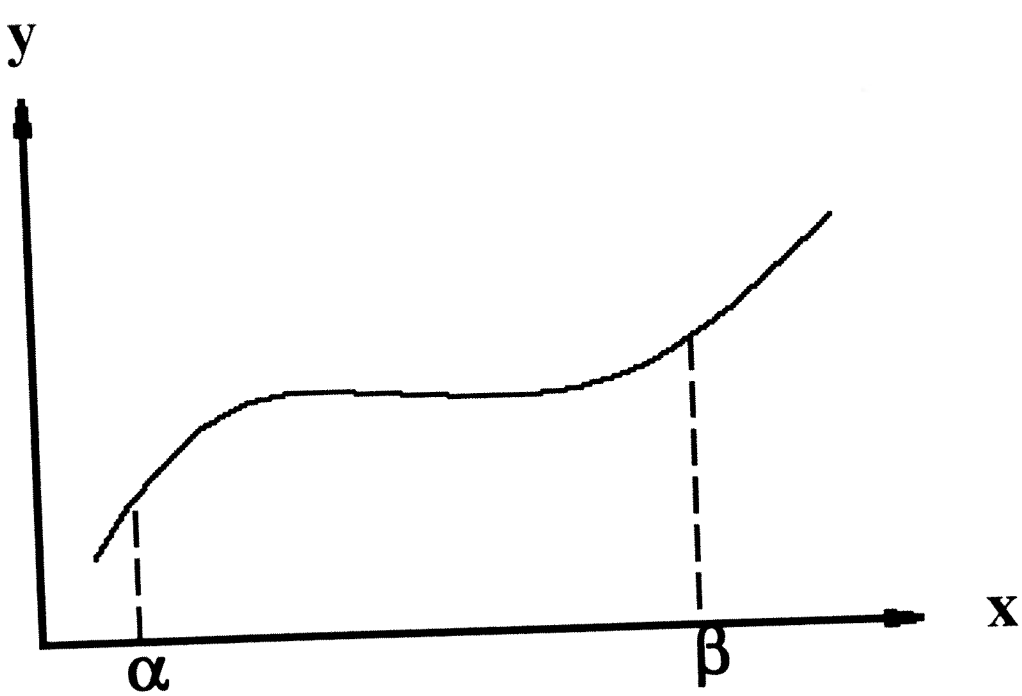
Once you have the linkage designed, you can scale it up or down and you can tighten the setscrews holding the input and output pointers on their shafts at any fixed offset angle that meets your fancy. You can even stand the whole linkage box on its end and it will still generate the same functional relationship between the input and the output!



Of course, it is possible you may have an actual function you want to generate as in:

$$y = f(x)$$

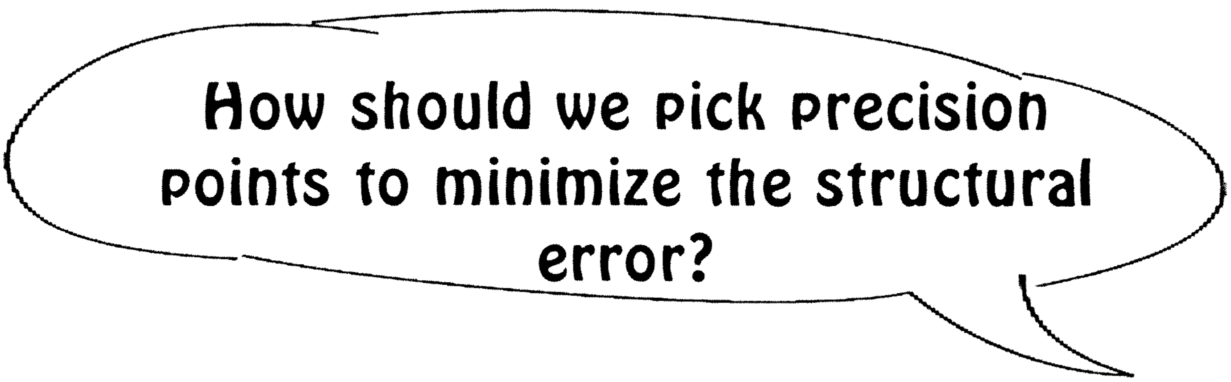
$$\alpha \leq x \leq \beta$$



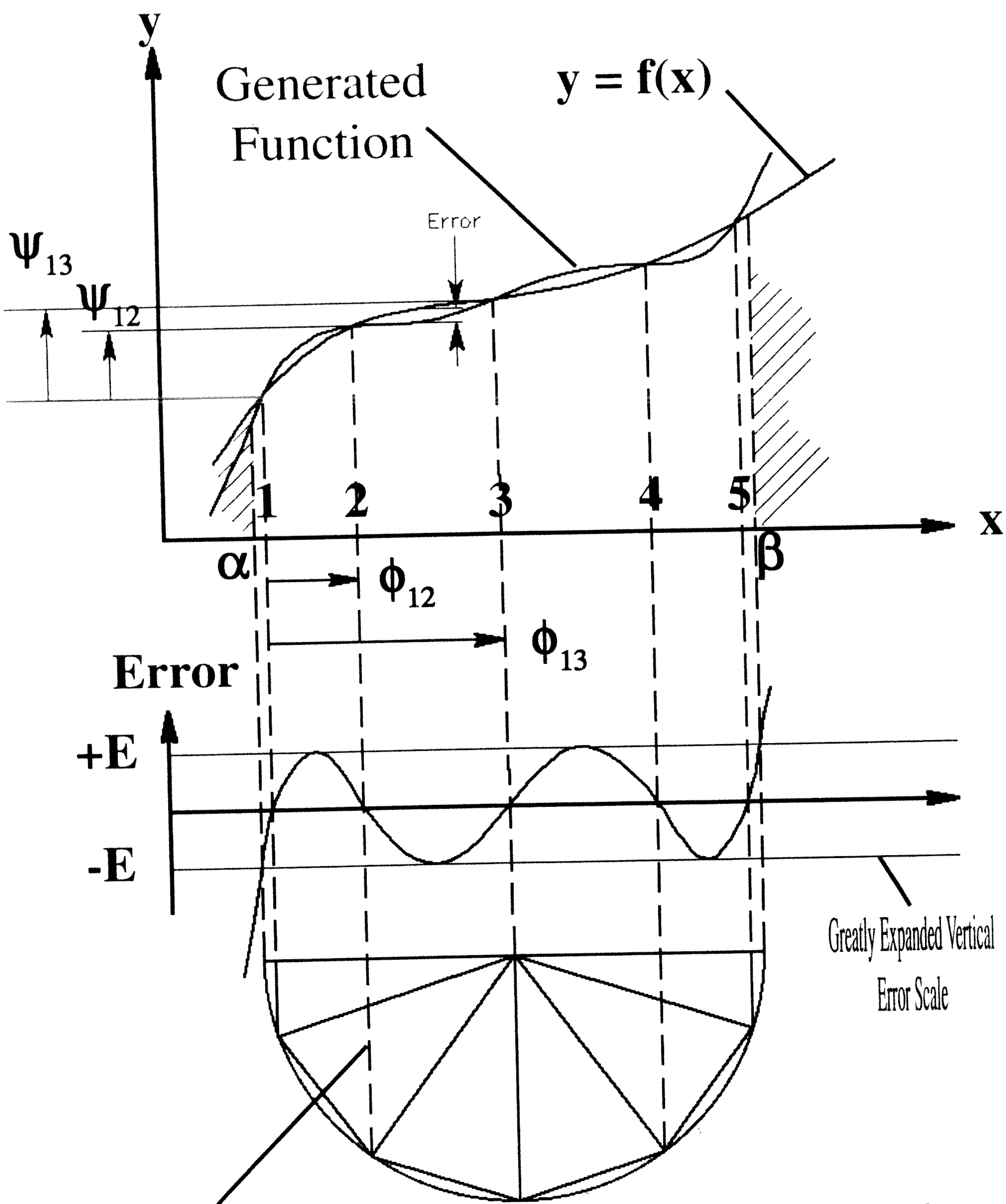
Once again, you can let ϕ be the linear analog of x and ψ be the linear analog of y and you can pick the scale factors $\mathbf{R}_\phi$ and $\mathbf{R}_\psi$ just the way you did before.

Given the limited number of adjustable parameters in a four-bar linkage, we will only be able to exactly match the desired function at five precision points. We will need to accept a built-in structural error in between those precision points but we can take steps to minimize that structural error. (In fact, if the function is well-behaved, we might not even be able to see the error but in theory it exists even if it is so miniscule that it doesn't matter to our application.)

This structural error is not to be confused with errors due to manufacturing tolerances or slop in the joints. It exists even if all the parts were perfectly made to nano-technology tolerances. It is hard-wired into the mechanism by the laws of mathematics. Trigonometry doesn't bow to any technology!



In a memoir on straight-line mechanisms *Théorie des Méchanismes Connus Sous Le Nom de Parallélogrammes* Chebyshev introduced Chebyshev Polynomials. He showed that with N accuracy points the *best* (least maximum error) situation occurs when the accuracy points are spaced so that the error reaches equal and opposite extreme values N + 1 times in the interval of interest. So with five accuracy points we'd want the structural error to look something like this:



Precision points are projections of the vertices of a 10 sided polygon inscribed on the interval

To construct the locations of the precision points first lay out a circle whose diameter runs from α to β on the x axis. Then inscribe a polygon with 2N sides in that circle. (Be sure that the sides of the polygon near α and β are perpendicular to the x axis.)

The best first-try spacing for the N precision points is given by the projection of the locations of the polygon's vertices. Notice that the first and last precision points *aren't* located at α or at β !

Using these precision points the structural error curve will wobble back and forth close to the desired ideal function. In between each of the given precision points the error will take on values that are approximately equal and opposite in sign. The error will also hit its maximum value just at the extreme ends of the desired interval. In order to illustrate the concept the error scale on the preceeding figure has been grossly exaggerated so you can actually see the error.

If the function being synthesized was a Chebyshev polynomial and it was being approximated by a lower order Chebyshev polynomial, this would be the ideal spacing but in practice the spacing might need to be fudged slightly if you wanted to perfect the spacing for a general function.

The important thing to note is that the linkage is synthesized at its first precision point (the reference position) and not at the start of the interval. It must be run back to get it into the actual starting position at α.